University of Illinois Spring 2024

ECE 313: Exam II Conflict 2

1. (a) Let X denote the number of cans observed with sugar content higher than 35 grams out of the 3 samples. Let H_0 denote the hypothesis that the company's claim is true, and H_1 denote the hypothesis that the consumer advocacy group's hypothesis is true. If H_0 is true, $X \sim \text{Bi}(3, \frac{1}{3})$; otherwise, if H_1 is true, $X \sim \text{Bi}(3, \frac{1}{2})$. The likelihood matrix is then:

	$X = 0$	$X = 1$	$X = 2$	$X = 3$
H_0	8/27	12/27	6/27	1/27
H_1	1/8	3/8	3/8	1/8

The ML decision rule is thus to declare H_0 if $X = 0, 1$ and declare H_1 if $X = 2, 3$.

(b) According to your definition of H_0 and H_1, your $p_{false\,alarm}$ may equal the value of p_{miss} in this solution, and vice versa. As long as your definitions are consistent, however, your p_e should be the same.

$$p_{false\,alarm} = P(declare\, H_1|H_0) \tag{1}$$
$$= P(X = 2, 3|H_0) \tag{2}$$
$$= \frac{7}{27} \tag{3}$$

$$p_{miss} = P(declare\, H_0|H_1) \tag{4}$$
$$= P(X = 0, 1|H_0) \tag{5}$$
$$= \frac{1}{2} \tag{6}$$

$$p_e = p_{false\,alarm} \cdot \frac{1}{4} + p_{miss} \cdot \frac{3}{4} \tag{7}$$
$$= \frac{7}{108} + \frac{3}{8} \tag{8}$$
$$= \frac{95}{216} \tag{9}$$

(c) The MAP joint probability matrix is then:

	$X=0$	$X=1$	$X=2$	$X=3$
H_0	8/108	12/108	6/108	1/108
H_1	3/32	9/32	9/32	3/32

Therefore H_1 is always chosen.

(d)

$$p_{false\,alarm} = P(declare\,H_1|H_0) \tag{10}$$
$$= P(X=0,1,2,3|H_0) \tag{11}$$
$$= 1 \tag{12}$$

$$p_{miss} = P(declare\,H_0|H_1) \tag{13}$$
$$= 0 \tag{14}$$

$$p_e = p_{false\,alarm}\cdot\frac{1}{4} + p_{miss}\cdot\frac{3}{4} \tag{15}$$
$$= \frac{1}{4} \tag{16}$$

Note that $\frac{1}{4} < \frac{95}{216}$.

2. (a)

$$P(N_{60} = 0) = \frac{e^{-60\lambda}(60\lambda)^0}{0!}$$
$$= e^{-60}$$

(b) Due to the independent increment property,

$$P(N_{120} - N_{60} = 1 | N_{60} = 10) = P(N_{120} - N_{60} = 1)$$
$$= \frac{e^{-60\lambda}(60\lambda)^1}{1!}$$
$$= 60e^{-60}$$

(c) Let A denote the event that the server encountered exactly 1 arrival in both of the two time intervals $[0, 2]$ and $[1, 4]$ seconds. There are two ways to meet such requirements:

	$[0, 1)$	$[1, 2]$	$(2, 4]$
Case #1	1	0	1
Case #2	0	1	0

$$P(A) = \frac{e^{-\lambda}\lambda^1}{1!}\frac{e^{-\lambda}\lambda^0}{0!}\frac{e^{-2\lambda}(2\lambda)^1}{1!} + \frac{e^{-\lambda}\lambda^0}{0!}\frac{e^{-\lambda}\lambda^1}{1!}\frac{e^{-2\lambda}(2\lambda)^0}{0!}$$
$$= 3e^{-4}$$

3. (a) We have

$$\text{(number of times odd number shows)} = 100 - X.$$

Therefore

$$A = \{X \leq 100 - X + 4\} = \{X \leq 52\}.$$

Since an even number (success) occurs with probability 3/6 on every die roll, $X \sim \text{Binom}(100, 0.5)$.

(b) Since X has the binomial distribution with parameters $n = 100$ and $p = 0.5$, therefore

$$P(A) = P\{X \leq 52\} = \sum_{k=0}^{52} \binom{100}{k} (0.5)^k (0.5)^{100-k} = (0.5)^{100} \sum_{k=0}^{52} \binom{100}{k}.$$

(c) Note that $E[X] = 50$, $\text{Var}(X) = (100)p(1-p) = 25$, and $\sigma = \sqrt{\text{Var}(X)} = 5$. Therefore

$$\begin{aligned} P(A) = P\{X \leq 52\} &\approx P\{\tilde{X} \leq 52.5\} \\ &= P\left\{\frac{\tilde{X} - 50}{5} \leq \frac{52.5 - 50}{5}\right\} \\ &= \Phi\left(\frac{52.5 - 50}{5}\right) = \Phi(0.5). \end{aligned}$$

where the approximation comes from Gaussian approximation with continuity correction.

(d) We have

$$\begin{aligned} P(B) &= P\{X = 55\} = P\{54.5 < X < 55.5\} \\ &\approx P\{54.5 < \tilde{X} < 55.5\} \\ &= P\left\{\frac{54.5 - 50}{5} < \frac{\tilde{X} - 50}{5} < \frac{55.5 - 50}{5}\right\} \\ &= Q\left(\frac{54.5 - 50}{5}\right) - Q\left(\frac{55.5 - 50}{5}\right) = Q(0.9) - Q(1.1). \end{aligned}$$

4. (a) A valid pdf should integrate to 1. For $c > 0$, $|c| = c$.

$$\begin{aligned} \int_0^{|c|} (cx + c^2) dx &= 1 \\ \frac{c|c|^2}{2} + c^2|c| &= 1 \\ \frac{3c^3}{2} &= 1 \\ c^3 &= \frac{2}{3} \\ c &= \sqrt[3]{\frac{2}{3}} \end{aligned}$$

(b) Since $X \geq 0$,

$$\begin{aligned} P(\frac{c}{4} < X^2 < \frac{c}{2}) &= P(\frac{\sqrt{c}}{2} < X < \frac{\sqrt{c}}{\sqrt{2}}) \\ &= \int_{\frac{\sqrt{c}}{2}}^{\frac{\sqrt{c}}{\sqrt{2}}} (cx + c^2) dx \\ &= \frac{cx^2}{2} + c^2 x \Big|_{\frac{\sqrt{c}}{2}}^{\frac{\sqrt{c}}{\sqrt{2}}} \\ &= \frac{c^2}{4} + \frac{c^{\frac{5}{2}}}{\sqrt{2}} - \frac{c^2}{8} - \frac{c^{\frac{5}{2}}}{2} \\ &= \frac{c^2}{8} + (\sqrt{2} - 1)\frac{c^{\frac{5}{2}}}{2} \end{aligned}$$