University of Illinois **Spring 2024**

ECE 313: Exam I

Monday, February 26, 2024
7:00 p.m. — 8:15 p.m.

1. (a) If we use (H, H, T), for example, to denote the case of "first toss head, second toss head, and third toss tail", then the sample space can be expressed as

$$\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$$

and each element is equally likely to occur. We then have

$$P(A_{12}) = P(A_{13}) = P(A_{23}) = \frac{4}{8} = \frac{1}{2}$$

and

$$\begin{aligned} P(A_{12}A_{13}) &= P(A_{12}A_{23}) = P(A_{13}A_{23}) \\ &= P(\text{all 3 tosses produce same outcome}) = \frac{2}{8} = \frac{1}{4} \end{aligned}$$

Therefore,

$$\begin{aligned} P(A_{12}A_{13}) &= P(A_{12})P(A_{13}) \\ P(A_{12}A_{23}) &= P(A_{12})P(A_{23}) \\ P(A_{13}A_{23}) &= P(A_{13})P(A_{23}) \end{aligned}$$

and thus these events are pairwise independent.

(b) As we have found that they are pairwise independent in (a), it remains to check whether

$$P(A_{12}A_{13}A_{23}) = P(A_{12})P(A_{13})P(A_{23})$$

However,

$$\begin{aligned} P(A_{12}A_{13}A_{23}) &= P(\text{all 3 tosses produce same outcome}) \\ &= \frac{1}{4} \\ &\neq \frac{1}{8} = P(A_{12})P(A_{13})P(A_{23}) \end{aligned}$$

so these events are not independent.

2. (a) Given that $p_X(m) = 5p_X(0)$, thus $c_2 = 5c_1(\frac{1}{2})^0$ or $c_2 = 5c_1$
$p_X(k)$ is a pmf, thus we have:

$$\sum_k p_X(k) = 1$$
$$c_1(\frac{1}{2})^{-2} + c_1(\frac{1}{2})^{-1} + c_1(\frac{1}{2})^0 + c_2 = 1$$
$$7c_1 + 5c_2 = 1$$

By solving the systems of equation:

$$\begin{cases} c_2 = 5c_1 \\ 7c_1 + c_2 = 1 \end{cases}$$

We get $c_1 = \frac{1}{12}$ and $c_2 = \frac{5}{12}$.

(b) Given that $c_1 = \frac{1}{12}$ and $c_2 = \frac{5}{12}$, we have:

$$E[X] = \sum_k kp_X(k) = -2(\frac{1}{12})(\frac{1}{2})^{-2} + (-1)(\frac{1}{12})(\frac{1}{2})^{-1} + 0 + \frac{5m}{12} = -\frac{10}{12} + \frac{5m}{12}$$

By setting $E[X] = 0$, we get $m = 2$.

(c) With $c_1 = \frac{1}{12}, c_2 = \frac{5}{12}, m = 2$, and $\mathbb{E}[X] = 0$, we have $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] = 4(\frac{1}{12})(\frac{1}{2})^{-2} + 1(\frac{1}{12})(\frac{1}{2})^{-1} + 0 + 4 \cdot \frac{5}{12} = \frac{4}{3} + \frac{1}{6} + \frac{5}{3} = \frac{19}{6}$

(d) $\mathbf{Pr}\{|X| \geq 2\} = \mathbf{Pr}\{X \geq 2\} + \mathbf{Pr}\{X \leq -2\} = \mathbf{Pr}\{X = 2\} + \mathbf{Pr}\{X = -2\} = \frac{5}{12} + \frac{1}{12} \cdot 4 = \frac{3}{4}$
$\mathbb{E}[|X|] = 2(\frac{1}{12})(\frac{1}{2})^{-2} + (1)(\frac{1}{12})(\frac{1}{2})^{-1} + 0 + \frac{10}{12} = \frac{8}{12} + \frac{2}{12} + \frac{10}{12} = \frac{5}{3}$

3. (a) The expected number of times you win the $5 dollar prize can be modeled as the expected number of successes for a binomial distribution with parameters $n = 52$, $p = 10^{-2}$. Therefore this is equal to 0.52, or $\frac{13}{25}$.

The probability that you win the main prize exactly once can be modeled as $P(X = 1)$ for a binomial random variable X with parameters $n = 52$, $p = 10^{-6}$.

$$P(X = 1) = p_X(1) = \binom{52}{1}(10^{-6})^1(1 - 10^{-6})^{52-1}$$

(b) This can be modeled as the expected number of trials before your first success, or the expected value of a geometric distribution with parameter $p = 10^{-6}$. The expected value is therefore

$$\frac{1}{10^{-6}} = 10^6$$

(c) Let p be the probability of winning the \$5 prize and $E[X]$ be the expected number of tickets before we win two \$5 awards consecutively. To calculate the expectation, we consider the various events that can occur and their corresponding probabilities. We use the independence of outcomes of any pair of tickets.

- The first ticket is a loss: This happens with a probability $(1-p)$. Since we did not win with this ticket and each round is independent, the expected number of additional tickets needed before we win two \$5 prizes consecutively is still $E[X]$. Counting the one extra ticket, the total expected number of tickets is $E[X]+1$.
- The first ticket is a win:
  - The second ticket is a loss: A win followed by a loss happens with a probability $p(1-p)$. Since we did not win two \$5 prizes consecutively and each round is independent, the expected number of additional tickets before we win two \$5 prizes consecutively is still $E[X]$. Counting the two extra tickets, the total expected number of tickets is $E[X]+2$.
  - The second ticket is a win: A win followed by a win happens with a probability p^2. Since we win two \$5 prizes consecutively in this case, the expected number of tickets in this case is 2.

Using these events and their corresponding probabilities, we can write the expectation as

$$\begin{aligned} E[X] &= (E[X]+1)(1-p) + (E[X]+2)(p(1-p)) + 2p^2 \\ E[X] &= (1-p^2)E[X] + 1 + p \\ E[X] &= \frac{1+p}{p^2} = \frac{1}{p} + \frac{1}{p^2} \end{aligned}$$

Substituting $p = 10^{-2}$, we get $E[X] = 10,100$.

Exercise: Show that the expected number of tickets you need to purchase until you win n \$5 awards in consecutive order is

$$\sum_{k=1}^{n} \frac{1}{p^k}$$

Note that the random variable X that represents the number of rolls before we see two \$5 prizes consecutively is NOT a geometric random variable. To see this, consider a sequence $T_0, T_1, T_2, \ldots$ of tickets purchased. We will have $X = i+1$ if both T_i, T_{i+1} are wins and this is happening for the first time. However, the probability of both being wins depends on the outcome of events T_{i-1}, T_i being (Loss, Win) or (Win, Loss) or (Loss, Loss). The memoryless property is not valid here.

One way to resolve this problem is to consider a different random variable $\hat{X}$ such that the probability of $\hat{X} = i+1$ is

$$P(\hat{X} = i+1) = P(X = i+1 | X = i \text{ was not a consecutive win}).$$

The memoryless property holds for this new random variable. (One student correctly used this approach to answer the question.)

4. (a) The probability $P\{X = 13\} = 1/n$ since X takes each value $\mathcal{S}$ with equal probability and there are n elements in $\mathcal{S}$. Also, since $X = 13$ has been observed $2n + 1 \geq 13$. This implies that $n_{\mathrm{ML}} = 6$.

(b) Since the three observations are independent, if $2n + 1 \geq 13$,

$$P\{X_1 = 13, X_2 = 7, X_3 = 9\} = P\{X_1 = 13\} \cdot P\{X_2 = 7\} \cdot P\{X_3 = 9\} = \frac{1}{n^3}$$

Hence, a reasonable estimate would be $\hat{n}_{\mathrm{ML}} = 6$, as that would maximize $P\{X_1 = 13, X_2 = 7, X_3 = 9\}$.