University of Illinois **Spring 2024**

ECE 313: Exam I Conflict

Tuesday, February 27, 2024
7:00 p.m. — 8:15 p.m.

1. (a)

$$P(A) = \frac{3}{6} = \frac{1}{2}$$
$$P(B) = \frac{3}{6} = \frac{1}{2}$$
$$P(C) = \frac{4}{36} = \frac{1}{9}$$
$$P(ABC) = P(X = 3, Y = 6) = \frac{1}{36}$$

Therefore, it is true that $P(ABC) = P(A)P(B)P(C)$.

(b) No, since

$$P(AB) = P(X = 3) = \frac{1}{6} \neq \frac{1}{4} = P(A)P(B)$$

or

$$P(AC) = P(X = 3, Y = 6) = \frac{1}{36} \neq \frac{1}{18} = P(A)P(C)$$

or

$$\begin{aligned} P(BC) &= P(X = 3, Y = 6) + P(X = 4, Y = 5) + P(X = 5, Y = 4) \\ &= \frac{3}{36} \neq \frac{1}{18} = P(B)P(C) \end{aligned}$$

indicates that these events are not pairwise independent, hence not independent.

2. (a)

$$p_X(-2) + p_X(0) + p_X(m) + p_X(m+2) = 1 \tag{1}$$
$$9c_1 + c_1 + c_2 + c_2 = 1 \tag{2}$$
$$10c_1 + 2c_2 = 1 \tag{3}$$

Additionally,

$$p_X(m+2) = 5p_X(0) \tag{4}$$
$$c_2 = 5c_1 \tag{5}$$

Solving this system of equations gives

$$c_1 = \frac{1}{20} \tag{6}$$
$$c_2 = \frac{1}{4} \tag{7}$$

(b)

$$-2 \cdot \frac{1}{20}\frac{1}{3}^{-2} + 0 + \frac{m}{4} + \frac{m+2}{4} = 0 \tag{8}$$
$$-\frac{9}{10} + \frac{m}{2} + \frac{1}{2} = 0 \tag{9}$$
$$m = \frac{4}{5} \tag{10}$$

(c) We know $Var(X) = E[X^2] + E[X]^2 = E[X^2]$. So

$$E[X^2] = 4 \cdot \frac{1}{20} \left(\frac{1}{3}\right)^{-2} + 0 + \frac{\frac{16}{25}}{4} + \frac{(\frac{14}{5})^2}{4} \tag{11}$$

$$= \frac{9}{5} + \frac{4}{25} + \frac{196}{4 \cdot 25} \tag{12}$$

$$= \frac{45}{25} + \frac{4}{25} + \frac{49}{25} \tag{13}$$

$$= \frac{98}{25} \tag{14}$$

(d) $\mathbf{Pr}\{|X| \geq 2\} = \mathbf{Pr}\{X \geq 2\} + \mathbf{Pr}\{X \leq -2\} = \mathbf{Pr}\{X = \frac{14}{5}\} + \mathbf{Pr}\{X = -2\} = \frac{1}{4} + \frac{9}{20} = \frac{7}{10}$
$\mathbb{E}[|X|] = 2(\frac{1}{20})(\frac{1}{3})^{-2} + \frac{4}{5} \cdot \frac{1}{4} + \frac{14}{5} \cdot \frac{1}{4} = \frac{9}{10} + \frac{1}{5} + \frac{7}{10} = \frac{9+2+7}{10} = \frac{9}{5}$

3. (a) This condition is satisfied when only A or only B happens. For only A to happen, the event $A \cap B^c$ must happen. For only B to happen, the event $A^c \cap B$ must happen. Therefore

$$P((A \cap B^c) \cup (A^c \cap B)) = P(A \cap B^c) + P(A^c \cap B) - P(A \cap B^c \cap A^c \cap B)) \tag{15}$$

$$= P(A \cap B^c) + P(A^c \cap B) \tag{16}$$

Alternatively, this can be expressed as a complement of another event. For only one event to happen, neither $A \cap B$ nor $A^c \cap B^c$ can happen.

$$P(((A \cap B) \cup (A^c \cap B^c))^c) = 1 - P((A \cap B) \cup (A^c \cap B^c)) \tag{17}$$

$$= 1 - (P(A \cap B) + P(A^c \cap B^c)) \tag{18}$$

$$= 1 - P(A \cap B) - P(A^c \cap B^c) \tag{19}$$

$$= 1 - P(A \cap B) - (1 - P(A \cup B)) \tag{20}$$

$$= P(A \cup B) - P(A \cap B) \tag{21}$$

(b) This can be represented as a binomial distribution where heads is a success and tails is a failure.
When n is even, we must have $\frac{n}{2}$ successes and $\frac{n}{2}$ failures. Therefore the probability is defined by the pmf to be

$$\binom{n}{\frac{n}{2}}(\frac{1}{4})^{\frac{n}{2}}(\frac{3}{4})^{\frac{n}{2}}$$

When n is odd, it is impossible to have an equal number of heads and tails.

(c) Since X and Y are non-negative random variables upper bounded by 10, we know $0 \leq X \leq 10$, $0 \leq Y \leq 10$, and thus $0 \leq X^2 \leq 100$, $0 \leq Y^2 \leq 100$. Besides, $X^2 + Y^2 \leq 20$ indicates $(100 - X^2) + (100 - Y^2) \geq 180$, so it is convenient to define two new random variables: $X' = 100 - X^2$ and $Y' = 100 - Y^2$. Note that X' and Y' are also non-negative random variables, so

$$
\begin{aligned}
P(X^2 + Y^2 \leq 20) &= P(X' + Y' \geq 180) && (22)\\
&\leq \frac{E[X' + Y']}{180} && (23)\\
&= \frac{E[X']}{180} + \frac{E[Y']}{180} && (24)\\
&= \frac{E[100 - X^2]}{180} + \frac{E[100 - Y^2]}{180} && (25)\\
&= \frac{100 - (E[X])^2 - \mathrm{Var}(X)}{180} + \frac{100 - (E[Y])^2 - \mathrm{Var}(Y)}{180} && (26)\\
&= \frac{100 - 4 - 1}{180} + \frac{100 - 4 - 2}{180} && (27)\\
&= \frac{21}{20} && (28)
\end{aligned}
$$

4. (a) $\hat{n}_{ML} = 5$. If it were lower, then 17 would not be a realizable value. If it were higher, then the probability of $X = 17$ would decrease.

(b)

$$P(X = 2 \cup X = 3) = P(X = 2) + P(X = 3) \tag{29}$$
$$= p(1-p) + p(1-p)^2 \tag{30}$$
$$= p - p^2 + p - 2p^2 + p^3 \tag{31}$$
$$= p^3 - 3p^2 + 2p \tag{32}$$

We need to maximize this sum, so we can derive and find the extrema.

$$(p^3 - 3p^2 + 2p)\frac{d}{dp} = 3p^2 - 6p + 2 \tag{33}$$
$$\tag{34}$$

With the quadratic formula, we can find that this polynomial equals zero when $p = \frac{3 \pm \sqrt{3}}{3}$. We know $\frac{3+\sqrt{3}}{3}$ is not a valid probability, so $\frac{3-\sqrt{3}}{3}$ is our extrema. We can also check for concavity. The second derivative is $6p - 6$, so as long as $p < 1$ the second derivative is negative and the function is concave down. Therefore our extrema is a maximum, and we can say that

$$\hat{p} = \frac{3 - \sqrt{3}}{3}$$