University of Illinois **Spring 2024**

ECE 313: Problem Set 12: Solutions

Due: Wednesday, May 1 at 7:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.5 - 4.11.

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. **Please write down your work and derivations. An answer without justification as of how it is found will not be accepted.** You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME AS IT APPEARS ON Canvas
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Correlation and covariance]**

   (a)
$$f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x) = xe^{-xy} \cdot 1$$
$$E[Y] = E[E[Y|X]] = E[\frac{1}{X}] = \ln(2)$$

$$\begin{aligned}
\text{Cov}(X,Y) &= E[XY] - E[X]E[Y] && (1)\\
&= \int_1^2 \int_0^\infty xy f_{X,Y}(x,y)dydx - 1.5 \cdot \ln(2) && (2)\\
&= \int_1^2 x \int_0^\infty yxe^{-xy}dydx - 1.5 \cdot \ln(2) && (3)\\
&= 1 - 1.5\ln(2) && (4)
\end{aligned}$$

   (b) The support of Z is $[1,4]$, so we need to find the CDF on that range. Suppose $c \in [1,4]$

$$\begin{aligned}
P(Z \leq c) &= P(X^2 \leq c) && (5)\\
&= P(-\sqrt{c} \leq X \leq \sqrt{c}) && (6)\\
&= P(X \leq \sqrt{c}) && (7)\\
&= \sqrt{c} - 1 && (8)
\end{aligned}$$

So,

$$F_Z(c) = \begin{cases} 0 & c < 1 \\ \sqrt{c} - 1 & 1 < c < 4 \\ 1 & \text{otherwise} \end{cases}$$

Deriving this shows that

$$f_Z(c) = \begin{cases} \frac{1}{2\sqrt{c}} & 1 < c < 4 \\ 0 & \text{otherwise} \end{cases}$$

The resultant expected value is

$$\begin{aligned} E[Z] &= \int_1^4 z \frac{1}{2\sqrt{z}} dz && (9) \\ &= \frac{1}{2} \int_1^4 \sqrt{z} dz && (10) \\ &= \frac{1}{2} (\frac{2}{3} z^{\frac{3}{2}})|_1^4 && (11) \\ &= \frac{7}{3} && (12) \end{aligned}$$

(c)

$$\begin{aligned} \text{Cov}(X, Z) &= \text{Cov}(X, X^2) && (13) \\ &= E[X^3] - E[X]E[X^2] && (14) \\ &= (4 - \frac{1}{4}) - (\frac{3}{2})(\frac{8}{3} - \frac{1}{3}) && (15) \\ &= \frac{1}{4} && (16) \end{aligned}$$

2. **[Joint pdfs of functions of random variables]**

(a) Note that the two random variables are independent. Let us represent the joint pdf of (W, Z) as $f_{W,Z}(c, d)$. The support of Z is clearly the set of nonnegative numbers. Because $W = \sqrt{Z} + 3X$, and $P(X \geq 0) = 1$, we can find that $P(W \geq \sqrt{Z}) = 1$. Therefore, we can start our pdf by writing

$$f_{W,Z}(c, d) = \begin{cases} ? & 0 \leq \sqrt{d} < c \\ 0 & \text{otherwise} \end{cases}$$

Referring to section 4.7.2 in the lecture notes, we can solve this problem without finding $P(W < c, Z < d)$ for arbitrary one-to-one functions of random variables by finding the Jacobian of the function and the inverses of the one-to-one functions. We know that $d = v^2$, and d must be positive, so $v = \sqrt{d}$. We also know that $c = \sqrt{d} + 3u$, so $u = \frac{c - \sqrt{d}}{3}$. Assuming our vector is of the form $\begin{pmatrix} W \\ Z \end{pmatrix}$, our Jacobian is

$$\begin{bmatrix} 3 & 1 \\ 0 & 2v \end{bmatrix}$$

Assume that $0 \leq \sqrt{d} \leq c$. Because X and Y are independent,

$$f_{W,Z}(c,d) = \frac{1}{|\det(\text{Jacobian})|} f_{X,Y}(\frac{c-\sqrt{d}}{3}, \sqrt{d}) \tag{17}$$
$$= \frac{1}{6\sqrt{d}} f_X(\frac{c-\sqrt{d}}{3}) f_Y(\sqrt{d}) \tag{18}$$
$$= \frac{1}{6\sqrt{d}} \exp(\frac{\sqrt{d}-c}{3}) \exp(-\sqrt{d}) \tag{19}$$
$$= \frac{1}{6\sqrt{d}} \exp(-(\frac{c+2\sqrt{d}}{3})) \tag{20}$$

So,

$$f_{W,Z}(c,d) = \begin{cases} \frac{1}{6\sqrt{d}} \exp(-(\frac{c+2\sqrt{d}}{3})) & 0 \leq \sqrt{d} < c \\ 0 & \text{otherwise} \end{cases}$$

(b) Because Z is the product of two nonnegative real numbers, the support of Z is the set of nonnegative real numbers.

$$P(Z \leq c) = P(XY \leq c) \tag{21}$$
$$= P(X \leq \frac{c}{Y}) \tag{22}$$
$$= \int_0^\infty f_Y(v) \int_0^{\frac{c}{v}} f_X(u) du dv \tag{23}$$
$$\tag{24}$$

Because the PDF of Z is the derivative of the CDF of Z, we can use the second fundamental theorem of calculus to simplify.

$$f_Z(c) = \frac{d}{dc} \int_0^\infty f_Y(v) \int_0^{\frac{c}{v}} f_X(u) du dv \tag{25}$$
$$= \int_0^\infty \exp(-v - \frac{c}{v}) \frac{1}{v} dv \tag{26}$$

Alt attempt(check):

$$\begin{aligned} F_Z(c) &= P(XY \leq c) \\ &= \int_0^\infty P(XY \leq c | Y = y) f_Y(y) dy \\ &= \int_0^\infty P(X \leq c/y | Y = y) f_Y(y) dy \\ &= \int_0^\infty F_X(c/y) f_Y(y) dy \\ &= \int_0^\infty (1 - e^{-c/y}) e^{-y} dy \\ &= \int_0^\infty (e^{-y} - e^{-c/y - y}) dy \end{aligned}$$

$$
\begin{aligned}
f_Z(c) &= \frac{d}{dc}\int_0^\infty (e^{-y} - e^{-c/y-y})dy \\
&= \frac{d}{dc}\int_0^\infty e^{-y}dy - \frac{d}{dc}\int_0^\infty e^{-c/y-y}dy \\
&= -\frac{d}{dc}\int_0^\infty e^{-c/y-y}dy \\
&= -\int_0^\infty e^{-y}\frac{d}{dc}e^{-c/y}dy \\
&= \int_0^\infty \frac{1}{y}e^{-y-c/y}dy
\end{aligned}
$$

3. **[Order statistics]**

(a) Refer to course notes, example 4.7.8. The joint pdf of Z and W is

$$
f_{W,Z}(w,z) = \begin{cases} f_{XY}(w,z) + f_{XY}(z,w) & w < z \\ 0 & \text{otherwise} \end{cases}
$$

Due to independence of X and Y, we have $f_{XY}(x,y) = f_X(x)f_Y(y)$, so the joint PDF can be rewrited as:

$$
f_{W,Z}(w,z) = \begin{cases} f_X(w)f_Y(z) + f_X(z)f_Y(w) & w < z \\ 0 & \text{otherwise} \end{cases}
$$

(b) Let $T = Z + W$. Note that $Z + W = X + Y$. From here, we can find the pdf as the sum of X and Y directly.

$$
\begin{aligned}
f_T(t) &= \frac{d}{dt}P(X + Y \leq t) && (27) \\
&= \frac{d}{dt}P(X \leq t - Y) && (28) \\
&= \frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{t-y} f_{X,Y}(x,y)dxdy && (29) \\
&= \int_{-\infty}^{\infty} f_{X,Y}(t-y,y)dy && (30)
\end{aligned}
$$

(c) Now, the supports of Z and W is the first quadrant in the cartesian coordinate plane. As a result, the joint pdf $f_{X,Y}(w,z) \neq 0$ when $0 < w, z$. Following our answer for part 3a,

$$
f_{W,Z}(w,z) = \begin{cases} 2e^{-w-z} & 0 < w < z \\ 0 & \text{otherwise} \end{cases}
$$

So the marginals are

$$f_W(w) = \int_w^\infty 2e^{-w-z} dz$$
$$= 2e^{-2w}, \quad w > 0$$
$$f_Z(z) = \int_0^z 2e^{-w-z} dw$$
$$= 2e^{-z} - 2e^{-2z}, \quad z > 0$$

4. **[More on covariances]**

(a)

$$\begin{aligned}\mathrm{Var}(2X+3Y) &= \mathrm{Cov}(2X+3Y, 2X+3Y) && (31)\\ &= \mathrm{Var}(2X) + 2\,\mathrm{Cov}(2X, 3Y) + \mathrm{Var}(3Y) && (32)\\ &= 4\,\mathrm{Var}(X) + 9\,\mathrm{Var}(Y) = 31 && (33)\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathrm{Cov}(X+Y, X+Y) && (34)\\ &= \mathrm{Var}(X) + 2\,\mathrm{Cov}(X, Y) + \mathrm{Var}(Y) && (35)\\ &= \mathrm{Var}(X) + \mathrm{Var}(Y) = 4 && (36)\end{aligned}$$

Solving this system of equations gives

$$\mathrm{Var}(X) = 1$$

$$\mathrm{Var}(Y) = 3$$

(b)

$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathrm{Var}(X-Y) && (37)\\ \mathrm{Var}(X) + 2\,\mathrm{Cov}(X,Y) + \mathrm{Var}(Y) &= \mathrm{Var}(X) - 2\,\mathrm{Cov}(X,Y) + \mathrm{Var}(Y) && (38)\\ \mathrm{Cov}(X,Y) &= -\,\mathrm{Cov}(X,Y) && (39)\end{aligned}$$

The only way this can be true is if the covariances are 0, so X and Y are uncorrelated.

(c) Not necessarily. Let Y be a uniform random variable distributed on $[0,1]$, and let X be a uniform random variable distributed on $[0,\sqrt{2}]$. The variance of X is $\frac{2}{12}$, and the variance of Y is $\frac{1}{12}$, so the condition is satisfied.

Then, if X and Y are independent, then they are definitely uncorrelated. However, if $X = \sqrt{2}Y$:

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathrm{Cov}(\sqrt{2}Y, Y) && (40)\\ &= \sqrt{2}(E[Y^2] - E[Y]^2) && (41)\\ &= \sqrt{2}\,\mathrm{Var}(Y) && (42)\\ &\neq 0 && (43)\end{aligned}$$

5. **[Law of Large Numbers and Central Limit Theorem]**

(a) By Chebychev inequality,

$$P\left\{|\frac{S_n}{n} - \mu_x| \geq 0.2\mu_x\right\} \leq \frac{\sigma_x^2}{0.04n\mu_x^2}$$
$$\implies P\left\{|\frac{S_n}{n} - \mu_x| < 0.2\mu_x\right\} \geq 1 - \frac{\sigma_x^2}{0.04n\mu_x^2}$$

As $\mu_x = \frac{1+2+3+4}{4} = 2.5$ and $\sigma_x^2 = \frac{1^2+2^2+3^2+4^2}{4} - \mu_x^2 = 1.25$, we have

$$1 - \frac{\sigma_x^2}{0.04n\mu_x^2} \geq 0.92$$
$$\implies 0.02n \geq 1.25$$
$$\implies n \geq \left\lceil \frac{1.25}{0.02} \right\rceil = 63$$

(b) By CLT, we have

$$P\left\{|\frac{S_n}{n} - \mu_x| \leq 0.2\mu_x\right\} = P\left\{|S_n - n\mu_x| \leq 0.2n\mu_x\right\}$$
$$= P\left\{|S_n - 2.5n| \leq 0.5n\right\} = P\left\{|\frac{S_n - 2.5n}{\sqrt{1.25n}}| \leq \frac{0.5n}{\sqrt{1.25n}}\right\}$$
$$\approx 1 - 2Q(\frac{0.5n}{\sqrt{1.25n}})$$

so we need

$$1 - 2Q(\frac{0.5n}{\sqrt{1.25n}}) \geq 0.92$$
$$\implies Q(\frac{0.5n}{\sqrt{1.25n}}) \leq 0.04$$
$$\implies \frac{0.5n}{\sqrt{1.25n}} \geq 1.75$$
$$\implies n \geq 16;$$

6. **[Jointly Gaussian Random Variables I]**

(a) To make X and W uncorrelated, we can set their covariance to 0. Note that

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{\mathrm{Cov}(X,Y)}{2} = \frac{1}{16}$$

Therefore

$$\mathrm{Cov}(X,Y) = \frac{1}{8}$$

$$\mathrm{Cov}(X,W) = \mathrm{Cov}(X, X + \alpha Y + \beta) \tag{44}$$
$$= \mathrm{Cov}(X, X + \alpha Y) \tag{45}$$
$$= \mathrm{Cov}(X,X) + \mathrm{Cov}(X, \alpha Y) \tag{46}$$
$$= 4 + \alpha\,\mathrm{Cov}(X,Y) \tag{47}$$
$$= 4 + \frac{\alpha}{8} \tag{48}$$

As a result, For X and W to be uncorrelated, $\alpha = -32$ and β is unconstrained. This does imply independence. Because $W + X = 2X + \alpha Y + \beta$, and because X, Y are jointly Gaussian, it is clear that $W + X$ produces a subset of $X + Y$. Therefore X, W are jointly Gaussian, and if they are uncorrelated they are independent.

Another way to see that X, W are jointly Gaussian is, suppose we have $Z = pX + qW$ for constants p, q, then

$$\begin{aligned} Z &= pX + q(X + \alpha Y + \beta) \\ &= (p+q)X + q\alpha Y + q\beta \end{aligned}$$

Since X, Y are jointly Gaussian, $(p+q)X + q\alpha Y$ is a Gaussian random variable. Adding a constant to a Gaussian random variable only shifts its mean, so Z, and thus all linear combinations of X and W, are Gaussian random variables.

(b)

$$\begin{aligned} E[Z] &= E[3X + 2Y + 4] && (49) \\ &= 3E[X] + 2E[Y] + 4 && (50) \\ &= 12 && (51) \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(Z) &= \mathrm{Var}(3X + 2Y + 4) && (52) \\ &= \mathrm{Var}(3X + 2Y) && (53) \\ &= E[(3X + 2Y)^2] - 64 && (54) \\ &= E[9X^2 + 12XY + 4Y^2] - 64 && (55) \\ &= 9(\mathrm{Var}(X) + E[X]^2) + 12(\mathrm{Cov}(X,Y) + E[X]E[Y]) + 4(\mathrm{Var}(Y) + E[Y]^2) - 64 && (56) \\ &= 9(4+4) + 12(\frac{1}{8} + 2) + 4(1+1) - 64 && (57) \\ &= \frac{83}{2} && (58) \\ & && (59) \end{aligned}$$

(c) Because Y and Z are jointly Gaussian due to the same reason as in (a), the unconstrained estimator is the same as the linear estimator. First, we need to find the covariance of Y and Z.

$$\begin{aligned} \mathrm{Cov}(Y, Z) &= \mathrm{Cov}(Y, 3X + 2Y + 4) && (60) \\ &= \mathrm{Cov}(Y, 3X + 2Y) && (61) \\ &= 3\,\mathrm{Cov}(Y, X) + 2\,\mathrm{Var}(Y) && (62) \\ &= \frac{3}{8} + 2 && (63) \\ &= \frac{19}{16} && (64) \end{aligned}$$

With that, we can solve for the linear estimator.

$$E[Y|Z=11] = \hat{E}[Y|Z=11] \tag{65}$$
$$= 1 + \frac{19}{16} \cdot \frac{2}{75}(11-12) \tag{66}$$
$$= 1 - \frac{19}{600} \tag{67}$$
$$= \frac{581}{600} \tag{68}$$

(d) We know that

$$\text{Var}(Y|Z=11) = E[Y^2|Z=11] - (E[Y|Z=11])^2$$

So if we can solve for $\text{Var}(Y|Z=11)$, we have an easy way to find the second moment. Intuitively, the variance is modeled as the average squared distance from a random variable to our estimate, it's expected value. Because this is equivalent to mean squared error, we can find that instead.

$$\text{Var}(Y|Z=11) = \text{Linear MMSE} \tag{69}$$
$$= 1 - (\frac{19}{16})^2 \cdot \frac{2}{75} \tag{70}$$
$$= 1 - (\frac{19}{80})^2 \cdot \frac{2}{3} \tag{71}$$

Therefore,

$$E[Y^2|Z=11] = \text{Linear MMSE} + (\frac{581}{600})^2 \tag{72}$$
$$= 1 - (\frac{19}{80})^2 \cdot \frac{2}{3} + \frac{581}{600} \tag{73}$$
$$= \frac{3707}{1920} \tag{74}$$

7. **[MMSE Estimation]**

(a)

$$\int_0^2 \int_0^2 c(x+y)dxdy = 1$$
$$\int_0^2 2c(y+1)dy = 1$$
$$8c = 1$$
$$c = \frac{1}{8}$$

(b) According to Eq.(4.30) in the course notes, we have to find $E[X|Y=1] = \int_{-\infty}^{\infty} x f_{X|Y}(x|1)dx = \int_0^2 x \frac{f_{XY}(x,1)}{f_Y(1)}dx$. The marginal pdf of Y can be obtained by

$$f_Y(y) = \int_0^2 \frac{1}{8}(x+y)dx$$
$$= \frac{1}{4}(y+1)$$

Therefore,

$$\begin{aligned}
E[X|Y=1] &= \int_0^2 x\frac{x+1}{8}(\frac{1}{4}(1+1))^{-1}dx \\
&= \int_0^2 \frac{x^2+x}{4}dx \\
&= \frac{7}{6}
\end{aligned}$$

(c) According to Eq.(4.35) in the course notes, we have to find $\widehat{E}[X|Y=y_0] = \mu_X + \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}(y_0 - \mu_Y)$.
The required values are

$$\begin{aligned}
\mu_X &= \int_0^2 x(\int_0^2 \frac{1}{8}(x+y)dy)dx \\
&= \int_0^2 \frac{x^2+x}{4}dx = \frac{7}{6} \\
\mu_Y &= \frac{7}{6} \qquad \text{[By Symmetry]} \\
\mathrm{Cov}(X,Y) &= E[XY] - \mu_X\mu_Y \\
&= \int_0^2\int_0^2 \frac{xy}{8}(x+y)dxdy - \frac{49}{36} \\
&= \int_0^2 (\frac{y}{3} + \frac{y^2}{4})dy - \frac{49}{36} \\
&= \frac{4}{3} - \frac{49}{36} = -\frac{1}{36} \\
\mathrm{Var}(Y) &= E[Y^2] - \mu_Y^2 \\
&= \int_0^2 \frac{y^2}{4}(y+1)dy - \frac{49}{36} \\
&= \frac{5}{3} - \frac{49}{36} = \frac{11}{36}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\widehat{E}[X|Y=y_0] &= \frac{7}{6} + \frac{-1/36}{11/36}(y_0 - \frac{7}{6}) \\
&= -\frac{1}{11}y_0 + \frac{14}{11}
\end{aligned}$$

8. **[Jointly Gaussian Random Variables II]**

(a) Since X and Y are independent Gaussian random variables, they are jointly Gaussian. Define $Z = 3X + 2Y$, then Z is a Gaussian random variable with

$$\mu_Z = 3 \times 1 + 2 \times 1 = 5$$

and

$$\mathrm{Var}(Z) = 9\,\mathrm{Var}(X) + 12\,\mathrm{Cov}(X,Y) + 4\,\mathrm{Var}(Y) = 9 \times 2 + 12 \times 0 + 4 \times 2 = 26$$

Therefore,

$$\begin{aligned} P(3X + 2Y + 1 \geq 3) &= P(Z \geq 2) \\ &= P(\frac{Z-5}{\sqrt{26}} \geq \frac{2-5}{\sqrt{26}}) \\ &= Q(-\frac{3}{\sqrt{26}}) \end{aligned}$$

(b)

$$\begin{aligned} P(X^2 + Y^2 \geq 1 + 2XY) &= P(X^2 + Y^2 - 2XY \geq 1) \\ &= P((X-Y)^2 \geq 1) \\ &= P(X - Y \geq 1) + P(X - Y \leq -1) \end{aligned}$$

Define $W = X - Y$, then W is a Gaussian random variable with

$$\mu_W = 1 \times 1 - 1 \times 1 = 0$$

and

$$\text{Var}(W) = \text{Var}(X) - 2\,\text{Cov}(X, Y) + \text{Var}(Y) = 2 + 2 \times 0 + 2 = 4$$

Therefore,

$$\begin{aligned} P(X^2 + Y^2 \geq 1 + 2XY) &= P(W \geq 1) + P(W \leq -1) \\ &= P(\frac{W}{2} \geq \frac{1}{2}) + P(\frac{W}{2} \leq -\frac{1}{2}) \\ &= Q(\frac{1}{2}) + \Phi(-\frac{1}{2}) \\ &= 2Q(\frac{1}{2}) \end{aligned}$$

9. **[Biased and unbiased estimators]**

(a) As $\sigma^2 = p(1-p)$, it is equivalent to determine if $E[\widehat{\sigma^2}] = p(1-p)$. As

$$\begin{aligned} E[\widehat{\sigma^2}] &= E[\hat{p}(1-\hat{p})] \\ &= E\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\left(1 - \frac{1}{n}\sum_{i=1}^{n} X_i\right)\right] \\ &= E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] - E\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] \end{aligned}$$

with

$$\begin{aligned} E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] &= \frac{1}{n}\sum_{i=1}^{n} E[X_i] \\ &= \frac{1}{n}(np) = p \end{aligned}$$

and (using $E[X_iX_j] = E[X_i]E[X_j]$ due to independence for all $i \neq j$)

$$
\begin{aligned}
E\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] &= \frac{1}{n^2}E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n} E[X_i^2] + 2\sum_{i\neq j} E[X_iX_j]\right) \\
&= \frac{1}{n^2}\left(n(p^2 + p(1-p)) + 2\binom{n}{2}p^2\right) \\
&= \frac{1}{n^2}(np + n^2p^2 - np^2) \\
&= p^2 + \frac{p(1-p)}{n}
\end{aligned}
$$

we find that

$$
E[\widehat{\sigma^2}] = p(1-p) - \frac{p(1-p)}{n} \neq p(1-p)
$$

Therefore, the estimator is biased, i.e. the statement is **false**.

(b) Similarly, we have to determine if $E[\widehat{p^2}] = p^2$. Using the results from (a),

$$
\begin{aligned}
E[\widehat{p^2}] &= E[\hat{p}^2] - \frac{1}{n-1}E[\hat{p}(1-\hat{p})] \\
&= E\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] - \frac{1}{n-1}E[\hat{p}(1-\hat{p})] \\
&= p^2 + \frac{p(1-p)}{n} - \frac{1}{n-1}\left(p(1-p) - \frac{p(1-p)}{n}\right) \\
&= p^2 + p(1-p)\left(\frac{1}{n} - \frac{1}{n-1} + \frac{1}{n(n-1)}\right) \\
&= p^2
\end{aligned}
$$

Therefore, the estimator is unbiased, i.e. the statement is **true**.

10. **[(Optional) Joint pmfs]**

Let A denote the discrete set. If $w, z \in A$ and $w < z$,

$$
\begin{aligned}
p_{WZ}(w,z) &= P(\min\{X,Y\} = w, \max\{X,Y\} = z) \\
&= P(X = w)P(Y = z) + P(X = z)P(Y = w) \\
&= p(w)p(z) + p(z)p(w) = 2p(w)p(z)
\end{aligned}
$$

Then, if $w, z \in A$ and $w = z$,

$$
\begin{aligned}
p_{WZ}(w,z) &= P(X = Y = w) \\
&= P(X = w)P(Y = w) \\
&= (p(w))^2
\end{aligned}
$$

Otherwise (i.e. if $w, z \in A$ but $w > z$, or at least one of w and $z \notin A$), $p_{WZ}(w, z) = 0$.

To sum up,

$$p_{WZ}(w,z) = \begin{cases} 2p(w)p(z) & w, z \in A,\ w < z, \\ (p(w))^2 & w, z \in A,\ w = z, \\ 0 & \text{otherwise.} \end{cases}$$

11. **[(Optional) More on LLN]**

Let $Z_i = h(X_i, Y_i)$ for $i = 1, 2, \ldots, n$, so $S_n = Z_1 + Z_2 + \cdots + Z_n$. Then

$$\mu_{Z_i} = E[X_i^2] + E[Y_i^2] = (0^2 + 1) + (1^2 + 2) = 4$$

Therefore, by Law of Large Numbers,

$$\lim_{n\to\infty} P(|\frac{S_n}{n} - 4| > \delta) = 0$$

for any $\delta > 0$. By plugging in $\delta = \frac{1}{2}$, we can conclude that the statement is **true**.

Extra Thinking: Find $\text{Var}(Z_i)$.

Reasoning:

$$\begin{aligned} \text{Var}(Z_i) &= \text{Var}(X_i^2) + \text{Var}(Y_i^2) \\ &= E[X_i^4] - (E[X_i^2])^2 + E[Y_i^4] - (E[Y_i^2])^2 \end{aligned}$$

Recall from Homework 9 that if $W_i \sim N(0, 1)$, we have $E[W_i^4] = \frac{4!}{2^2 \times 2!} = 3$ and $E[W_i^m] = 0$ for odd m. Rewriting $Y_i = \sqrt{2}W_i + 1$, we have

$$\begin{aligned} E[Y_i^4] &= E[(\sqrt{2}W_i + 1)^4] \\ &= 4E[W_i^4] + 8\sqrt{2}E[W_i^3] + 12E[W_i^2] + 4\sqrt{2}E[W_i] + 1 \\ &= 4 \times 3 + 0 + 12 \times 1 + 0 + 1 \\ &= 25 \end{aligned}$$

Since $X_i \sim N(0, 1)$ as well, $E[X_i^4] = 3$. Therefore,

$$\text{Var}(Z_i) = 3 - 1^2 + 25 - 3^2 = 18$$