University of Illinois **Spring 2024**

ECE 313: Problem Set 12: Problems and Solutions

Due: Wednesday, May 1 at 7:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.5 - 4.11.

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. **Please write down your work and derivations. An answer without justification as of how it is found will not be accepted.** You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME AS IT APPEARS ON Canvas
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SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Correlation and covariance]**
Suppose $X \sim \text{Unif}(1,2)$, and given $X = x$, Y is exponential with parameter $\lambda = x$.

   (a) Find the $\text{Cov}(X, Y)$.
   Solution:

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x) = xe^{-xy} \cdot 1$$

$$E[Y] = E[E[Y|X]] = E[\frac{1}{X}] = \ln(2)$$

$$\begin{aligned}
\text{Cov}(X,Y) &= E[XY] - E[X]E[Y] && (1)\\
&= \int_1^2 \int_0^\infty xyf_{X,Y}(x,y)dydx - 1.5 \cdot \ln(2) && (2)\\
&= \int_1^2 x \int_0^\infty yxe^{-xy}dydx - 1.5 \cdot \ln(2) && (3)\\
&= 1 - 1.5\ln(2) && (4)
\end{aligned}$$

   (b) Let $Z = X^2$. Find the pdf and expected value of Z.
   Solution: The support of Z is $[1, 4]$, so we need to find the CDF on that range. Suppose

$c \in [1, 4]$

$$\begin{aligned} P(Z \leq c) &= P(X^2 \leq c) && (5)\\ &= P(-\sqrt{c} \leq X \leq \sqrt{c}) && (6)\\ &= P(X \leq \sqrt{c}) && (7)\\ &= \sqrt{c} - 1 && (8) \end{aligned}$$

So,

$$F_Z(c) = \begin{cases} 0 & c < 1 \\ \sqrt{c} - 1 & 1 < c < 4 \\ 1 & \text{otherwise} \end{cases}$$

Deriving this shows that

$$f_Z(c) = \begin{cases} \frac{1}{2\sqrt{c}} & 1 < c < 4 \\ 0 & \text{otherwise} \end{cases}$$

The resultant expected value is

$$\begin{aligned} E[Z] &= \int_1^4 z \frac{1}{2\sqrt{z}} dz && (9)\\ &= \frac{1}{2} \int_1^4 \sqrt{z} dz && (10)\\ &= \frac{1}{2} (\frac{2}{3} z^{\frac{3}{2}})|_1^4 && (11)\\ &= \frac{7}{3} && (12) \end{aligned}$$

(c) Find $\text{Cov}(X, Z)$.
Solution:

$$\begin{aligned} \text{Cov}(X, Z) &= \text{Cov}(X, X^2) && (13)\\ &= E[X^3] - E[X]E[X^2] && (14)\\ &= (4 - \frac{1}{4}) - (\frac{3}{2})(\frac{8}{3} - \frac{1}{3}) && (15)\\ &= \frac{1}{4} && (16) \end{aligned}$$

2. **[Joint pdfs of functions of random variables]**
Let X and Y have joint pdf:

$$f_{X,Y}(u, v) = \begin{cases} e^{-u-v} & u, v \geq 0, \\ 0 & \text{else.} \end{cases}$$

(a) Find the joint pdf of (W, Z), where $W = Y + 3X$ and $Z = Y^2$.
Solution: Note that the two random variables are independent. Let us represent the joint pdf of (W, Z) as $f_{W,Z}(c, d)$. The support of Z is clearly the set of nonnegative

numbers. Because $W = \sqrt{Z}+3X$, and $P(X \geq 0) = 1$, we can find that $P(W \geq \sqrt{Z}) = 1$. Therefore, we can start our pdf by writing

$$f_{W,Z}(c,d) = \begin{cases} ? & 0 \leq \sqrt{d} < c \\ 0 & \text{otherwise} \end{cases}$$

Referring to section 4.7.2 in the lecture notes, we can solve this problem without finding $P(W < c, Z < d)$ for arbitrary one-to-one functions of random variables by finding the Jacobian of the function and the inverses of the one-to-one functions. We know that $d = v^2$, and d must be positive, so $v = \sqrt{d}$. We also know that $c = \sqrt{d}+3u$, so $u = \frac{c-\sqrt{d}}{3}$. Assuming our vector is of the form $\begin{pmatrix} W \\ Z \end{pmatrix}$, our Jacobian is

$$\begin{bmatrix} 3 & 1 \\ 0 & 2v \end{bmatrix}$$

Assume that $0 \leq \sqrt{d} \leq c$. Because X and Y are independent,

$$f_{W,Z}(c,d) = \frac{1}{|\det(\text{Jacobian})|} f_{X,Y}(\frac{c-\sqrt{d}}{3}, \sqrt{d}) \tag{17}$$

$$= \frac{1}{6\sqrt{d}} f_X(\frac{c-\sqrt{d}}{3}) f_Y(\sqrt{d}) \tag{18}$$

$$= \frac{1}{6\sqrt{d}} \exp(\frac{\sqrt{d}-c}{3}) \exp(-\sqrt{d}) \tag{19}$$

$$= \frac{1}{6\sqrt{d}} \exp(-(\frac{c+2\sqrt{d}}{3})) \tag{20}$$

So,

$$f_{W,Z}(c,d) = \begin{cases} \frac{1}{6\sqrt{d}} \exp(-(\frac{c+2\sqrt{d}}{3})) & 0 \leq \sqrt{d} < c \\ 0 & \text{otherwise} \end{cases}$$

(b) Find the pdf of $Z = XY$.

Solution: Because Z is the product of two nonnegative real numbers, the support of Z is the set of nonnegative real numbers.

$$P(Z \leq c) = P(XY \leq c) \tag{21}$$

$$= P(X \leq \frac{c}{Y}) \tag{22}$$

$$= \int_0^\infty f_Y(v) \int_0^{\frac{c}{v}} f_X(u) du dv \tag{23}$$

$$\tag{24}$$

Because the PDF of Z is the derivative of the CDF of Z, we can use the second fundamental theorem of calculus to simplify.

$$f_Z(c) = \frac{d}{dc} \int_0^\infty f_Y(v) \int_0^{\frac{c}{v}} f_X(u) du dv \tag{25}$$

$$= \int_0^\infty \exp(-v - \frac{c}{v}) \frac{1}{v} dv \tag{26}$$

Alt attempt(check):

$$\begin{aligned} F_Z(c) &= P(XY \le c) \\ &= \int_0^\infty P(XY \le c | Y = y) f_Y(y) dy \\ &= \int_0^\infty P(X \le c/y | Y = y) f_Y(y) dy \\ &= \int_0^\infty F_X(c/y) f_Y(y) dy \\ &= \int_0^\infty (1 - e^{-c/y}) e^{-y} dy \\ &= \int_0^\infty (e^{-y} - e^{-c/y - y}) dy \end{aligned}$$

$$\begin{aligned} f_Z(c) &= \frac{d}{dc} \int_0^\infty (e^{-y} - e^{-c/y-y}) dy \\ &= \frac{d}{dc} \int_0^\infty e^{-y} dy - \frac{d}{dc} \int_0^\infty e^{-c/y-y} dy \\ &= -\frac{d}{dc} \int_0^\infty e^{-c/y-y} dy \\ &= -\int_0^\infty e^{-y} \frac{d}{dc} e^{-c/y} dy \\ &= \int_0^\infty \frac{1}{y} e^{-y-c/y} dy \end{aligned}$$

3. **[Order statistics]**
Consider two random variables X and Y on the same probability space that are jointly continuous, with pdf $f_{X,Y}(x, y)$.

(a) Find an expression for the joint pdf of $Z = \max\{X, Y\}$ and $W = \min\{X, Y\}$. Rewrite the expression for the case that X and Y are independent, using their marginals.

Solution: Refer to course notes, example 4.7.8. The joint pdf of Z and W is

$$f_{W,Z}(w, z) = \begin{cases} f_{XY}(w, z) + f_{XY}(z, w) & w < z \\ 0 & \text{otherwise} \end{cases}$$

Due to independence of X and Y, we have $f_{XY}(x, y) = f_X(x) f_Y(y)$, so the joint PDF can be rewrited as:

$$f_{W,Z}(w, z) = \begin{cases} f_X(w) f_Y(z) + f_X(z) f_Y(w) & w < z \\ 0 & \text{otherwise} \end{cases}$$

(b) Write down the expression for the pdf of the sum $Z + W$ using $f_{X,Y}(x, y)$.

Solution: Let $T = Z + W$. Note that $Z + W = X + Y$. From here, we can find the pdf as the sum of X and Y directly.

$$f_T(t) = \frac{d}{dt}P(X + Y \leq t) \tag{27}$$
$$= \frac{d}{dt}P(X \leq t - Y) \tag{28}$$
$$= \frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{t-y} f_{X,Y}(x,y)dxdy \tag{29}$$
$$= \int_{-\infty}^{\infty} f_{X,Y}(t-y,y)dy \tag{30}$$

(c) Assume now that X and Y are independent exponential random variables with the same parameter $\lambda = 1$. Find the pdfs of Z and W.

Solution: Now, the supports of Z and W is the first quadrant in the cartesian coordinate plane. As a result, the joint pdf $f_{X,Y}(w,z) \neq 0$ when $0 < w, z$. Following our answer for part 3a,

$$f_{W,Z}(w,z) = \begin{cases} 2e^{-w-z} & 0 < w < z \\ 0 & \text{otherwise} \end{cases}$$

So the marginals are

$$f_W(w) = \int_w^{\infty} 2e^{-w-z}dz$$
$$= 2e^{-2w}, \quad w > 0$$
$$f_Z(z) = \int_0^z 2e^{-w-z}dw$$
$$= 2e^{-z} - 2e^{-2z}, \quad z > 0$$

4. **[More on covariances]**
Consider two random variables X and Y on the same probability space.

(a) Assuming that X and Y have zero means and are uncorrelated, find the variances of X and Y given that $\text{Var}(2X + 3Y) = 31$ and $\text{Var}(X + Y) = 4$.

Solution:

$$\text{Var}(2X+3Y) = \text{Cov}(2X+3Y, 2X+3Y) \tag{31}$$
$$= \text{Var}(2X) + 2\,\text{Cov}(2X, 3Y) + \text{Var}(3Y) \tag{32}$$
$$= 4\,\text{Var}(X) + 9\,\text{Var}(Y) = 31 \tag{33}$$

$$\text{Var}(X+Y) = \text{Cov}(X+Y, X+Y) \tag{34}$$
$$= \text{Var}(X) + 2\,\text{Cov}(X,Y) + \text{Var}(Y) \tag{35}$$
$$= \text{Var}(X) + \text{Var}(Y) = 4 \tag{36}$$

Solving this system of equations gives

$$\text{Var}(X) = 1$$

$$\text{Var}(Y) = 3$$

(b) Next, assume that you have no knowledge about whether X and Y are uncorrelated or not, and that the expectations of the variables are not necessarily zero. If $\mathrm{Var}(X+Y) = \mathrm{Var}(X-Y)$, are X and Y uncorrelated?

Solution:

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X-Y) \tag{37}$$
$$\mathrm{Var}(X) + 2\,\mathrm{Cov}(X,Y) + \mathrm{Var}(Y) = \mathrm{Var}(X) - 2\,\mathrm{Cov}(X,Y) + \mathrm{Var}(Y) \tag{38}$$
$$\mathrm{Cov}(X,Y) = -\,\mathrm{Cov}(X,Y) \tag{39}$$

The only way this can be true is if the covariances are 0, so X and Y are uncorrelated.

(c) If $\mathrm{Var}(X) = 2\,\mathrm{Var}(Y)$, do X and Y have to be uncorrelated? If so, justify your answer. If not, provide a counterexample.

Solution: Not necessarily. Let Y be a uniform random variable distributed on $[0,1]$, and let X be a uniform random variable distributed on $[0,\sqrt{2}]$. The variance of X is $\frac{2}{12}$, and the variance of Y is $\frac{1}{12}$, so the condition is satisfied.

Then, if X and Y are independent, then they are definitely uncorrelated. However, if $X = \sqrt{2}Y$:

$$\mathrm{Cov}(X,Y) = \mathrm{Cov}(\sqrt{2}Y, Y) \tag{40}$$
$$= \sqrt{2}(E[Y^2] - E[Y]^2) \tag{41}$$
$$= \sqrt{2}\,\mathrm{Var}(Y) \tag{42}$$
$$\neq 0 \tag{43}$$

5. **[Law of Large Numbers and Central Limit Theorem]**
A fair four-sided die is rolled n times. Let $S_n = X_1 + X_2 + \ldots + X_n$, where X_i is the number showing on the ith roll. Determine a condition on n such that the probability that the sample average $\frac{S_n}{n}$ is within 20% of the mean μ_X is greater than 0.92.

(a) Solve the problem using the form of the law of large numbers based on the Chebychev inequality.

Solution: By Chebychev inequality,

$$P\left\{|\frac{S_n}{n} - \mu_x| \geq 0.2\mu_x\right\} \leq \frac{\sigma_x^2}{0.04n\mu_x^2}$$
$$\Longrightarrow P\left\{|\frac{S_n}{n} - \mu_x| < 0.2\mu_x\right\} \geq 1 - \frac{\sigma_x^2}{0.04n\mu_x^2}$$

As $\mu_x = \frac{1+2+3+4}{4} = 2.5$ and $\sigma_x^2 = \frac{1^2+2^2+3^2+4^2}{4} - \mu_x^2 = 1.25$, we have

$$1 - \frac{\sigma_x^2}{0.04n\mu_x^2} \geq 0.92$$
$$\Longrightarrow 0.02n \geq 1.25$$
$$\Longrightarrow n \geq \left\lceil \frac{1.25}{0.02} \right\rceil = 63$$

(b) Solve the problem using the Gaussian approximation for S_n, which is suggested by the CLT. If you need to find $\mathcal{Q}(x)$ or $\Phi(x)$, round x to the nearest hundredth. (**Note**: Do not use the continuity correction for this question, because, unless $2.5n \pm (0.2)n\mu_X$ are integers, inserting the term 0.5 is not applicable).

Solution: By CLT, we have

$$\begin{aligned}
&P\left\{|\frac{S_n}{n} - \mu_x| \leq 0.2\mu_x\right\} = P\left\{|S_n - n\mu_x| \leq 0.2n\mu_x\right\} \\
=&P\left\{|S_n - 2.5n| \leq 0.5n\right\} = P\left\{|\frac{S_n - 2.5n}{\sqrt{1.25n}}| \leq \frac{0.5n}{\sqrt{1.25n}}\right\} \\
\approx&1 - 2Q(\frac{0.5n}{\sqrt{1.25n}})
\end{aligned}$$

so we need

$$\begin{aligned}
&1 - 2Q(\frac{0.5n}{\sqrt{1.25n}}) \geq 0.92 \\
\Longrightarrow\ & Q(\frac{0.5n}{\sqrt{1.25n}}) \leq 0.04 \\
\Longrightarrow\ & \frac{0.5n}{\sqrt{1.25n}} \geq 1.75 \\
\Longrightarrow\ & n \geq 16;
\end{aligned}$$

6. **[Jointly Gaussian Random Variables I]**
Suppose X and Y are jointly Gaussian with $\mu_X = 2$, $\mu_Y = 1$, $\sigma_X^2 = 4$, $\sigma_Y^2 = 1$, and $\rho_{XY} = \frac{1}{16}$.

   (a) Let $W = X + \alpha Y + \beta$.
   Find the values of α and β that make X and W uncorrelated. Will these values α and β make X and W independent? Explain why.
   Solution: To make X and W uncorrelated, we can set their covariance to 0. Note that

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{\mathrm{Cov}(X,Y)}{2} = \frac{1}{16}$$

   Therefore

$$\mathrm{Cov}(X,Y) = \frac{1}{8}$$

$$\begin{aligned}
\mathrm{Cov}(X,W) &= \mathrm{Cov}(X, X + \alpha Y + \beta) && (44) \\
&= \mathrm{Cov}(X, X + \alpha Y) && (45) \\
&= \mathrm{Cov}(X,X) + \mathrm{Cov}(X, \alpha Y) && (46) \\
&= 4 + \alpha\,\mathrm{Cov}(X,Y) && (47) \\
&= 4 + \frac{\alpha}{8} && (48)
\end{aligned}$$

   As a result, For X and W to be uncorrelated, $\alpha = -32$ and β is unconstrained. This does imply independence. Because $W + X = 2X + \alpha Y + \beta$, and because X, Y are jointly Gaussian, it is clear that $W + X$ produces a subset of $X + Y$. Therefore X, W are jointly Gaussian, and if they are uncorrelated they are independent.

Another way to see that X, W are jointly Gaussian is, suppose we have $Z = pX + qW$ for constants p, q, then

$$\begin{aligned} Z &= pX + q(X + \alpha Y + \beta) \\ &= (p+q)X + q\alpha Y + q\beta \end{aligned}$$

Since X, Y are jointly Gaussian, $(p+q)X + q\alpha Y$ is a Gaussian random variable. Adding a constant to a Gaussian random variable only shifts its mean, so Z, and thus all linear combinations of X and W, are Gaussian random variables.

(b) Let $Z = 3X + 2Y + 4$. Find the mean and variance of Z.

Solution:

$$\begin{aligned} E[Z] &= E[3X + 2Y + 4] && (49) \\ &= 3E[X] + 2E[Y] + 4 && (50) \\ &= 12 && (51) \end{aligned}$$

$$\begin{aligned} \text{Var}(Z) &= \text{Var}(3X + 2Y + 4) && (52) \\ &= \text{Var}(3X + 2Y) && (53) \\ &= \text{Var}(3X) + 2\,\text{Cov}(3X, 2Y) + \text{Var}(2Y) && (54) \\ &= 9\,\text{Var}(X) + 12\,\text{Cov}(X, Y) + 4\,\text{Var}(Y) && (55) \\ &= 36 + \frac{12}{8} + 4 && (56) \\ &= \frac{83}{2} && (57) \end{aligned}$$

(c) Find $E[Y|Z = 11]$.

Solution: Because Y and Z are jointly Gaussian due to the same reason as in (a), the unconstrained estimator is the same as the linear estimator. First, we need to find the covariance of Y and Z.

$$\begin{aligned} \text{Cov}(Y, Z) &= \text{Cov}(Y, 3X + 2Y + 4) && (58) \\ &= \text{Cov}(Y, 3X + 2Y) && (59) \\ &= 3\,\text{Cov}(Y, X) + 2\,\text{Var}(Y) && (60) \\ &= \frac{3}{8} + 2 && (61) \\ &= \frac{19}{8} && (62) \end{aligned}$$

With that, we can solve for the linear estimator.

$$\begin{aligned} E[Y|Z = 11] &= \hat{E}[Y|Z = 11] && (63) \\ &= 1 + \frac{19}{8} \cdot \frac{2}{83}(11 - 12) && (64) \\ &= 1 - \frac{19}{332} && (65) \\ &= \frac{313}{332} && (66) \end{aligned}$$

(d) Find $E[Y^2|Z=11]$.

Solution: We know that

$$\text{Var}(Y|Z=11) = E[Y^2|Z=11] - (E[Y|Z=11])^2$$

So if we can solve for $\text{Var}(Y|Z=11)$, we have an easy way to find the second moment. For jointly Gaussian random variables, it turns out that this variance is equivalent to mean squared error.

$$\begin{aligned}\text{Var}(Y|Z=11) &= \text{Linear MMSE} &(67)\\ &= 1-(\frac{19}{8})^2\cdot\frac{2}{83} &(68)\\ & &(69)\end{aligned}$$

Therefore,

$$\begin{aligned}E[Y^2|Z=11] &= \text{Linear MMSE} + (\frac{313}{332})^2 &(70)\\ &= 1-(\frac{19}{8})^2\cdot\frac{2}{83}+(\frac{313}{332})^2 &(71)\\ &= 1.7529 &(72)\end{aligned}$$

7. **[MMSE Estimation]**

Suppose X, Y have joint pmf

$$f_{XY}(x,y) = c(x+y), \qquad 0\le x\le 2,\ 0\le y\le 2$$

and 0 else.

(a) Find c.

Solution:

$$\begin{aligned}\int_0^2\int_0^2 c(x+y)dxdy &= 1\\ \int_0^2 2c(y+1)dy &= 1\\ 8c &= 1\\ c &= \frac{1}{8}\end{aligned}$$

(b) Find the unconstrained estimate of X given $Y=1$.

Solution: According to Eq.(4.30) in the course notes, we have to find $E[X|Y=1] = \int_{-\infty}^{\infty} x f_{X|Y}(x|1)dx = \int_0^2 x\frac{f_{XY}(x,1)}{f_Y(1)}dx$. The marginal pdf of Y can be obtained by

$$\begin{aligned}f_Y(y) &= \int_0^2 \frac{1}{8}(x+y)dx\\ &= \frac{1}{4}(y+1)\end{aligned}$$

Therefore,

$$\begin{aligned}
E[X|Y=1] &= \int_0^2 x\frac{x+1}{8}(\frac{1}{4}(1+1))^{-1}dx \\
&= \int_0^2 \frac{x^2+x}{4}dx \\
&= \frac{7}{6}
\end{aligned}$$

(c) Find the best linear estimator of X given $Y = y_0$.

Solution: According to Eq.(4.35) in the course notes, we have to find $\widehat{E}[X|Y=y_0] = \mu_X + \frac{\text{Cov}(X,Y)}{\text{Var}(Y)}(y_0 - \mu_Y)$.

The required values are

$$\begin{aligned}
\mu_X &= \int_0^2 x(\int_0^2 \frac{1}{8}(x+y)dy)dx \\
&= \int_0^2 \frac{x^2+x}{4}dx = \frac{7}{6} \\
\mu_Y &= \frac{7}{6} \qquad \text{[By Symmetry]} \\
\text{Cov}(X,Y) &= E[XY] - \mu_X\mu_Y \\
&= \int_0^2\int_0^2 \frac{xy}{8}(x+y)dxdy - \frac{49}{36} \\
&= \int_0^2 (\frac{y}{3} + \frac{y^2}{4})dy - \frac{49}{36} \\
&= \frac{4}{3} - \frac{49}{36} = -\frac{1}{36} \\
\text{Var}(Y) &= E[Y^2] - \mu_Y^2 \\
&= \int_0^2 \frac{y^2}{4}(y+1)dy - \frac{49}{36} \\
&= \frac{5}{3} - \frac{49}{36} = \frac{11}{36}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\widehat{E}[X|Y=y_0] &= \frac{7}{6} + \frac{-1/36}{11/36}(y_0 - \frac{7}{6}) \\
&= -\frac{1}{11}y_0 + \frac{14}{11}
\end{aligned}$$

8. **[Jointly Gaussian Random Variables II]**
Let X, Y be independent $N(1,2)$ random variables.

(a) Express $P(3X + 2Y + 1 \geq 3)$ in terms of the Q function.

Solution: Since X and Y are independent Gaussian random variables, they are jointly Gaussian. Define $Z = 3X + 2Y$, then Z is a Gaussian random variable with

$$\mu_Z = 3 \times 1 + 2 \times 1 = 5$$

and

$$\text{Var}(Z) = 9\,\text{Var}(X) + 12\,\text{Cov}(X,Y) + 4\,\text{Var}(Y) = 9 \times 2 + 12 \times 0 + 4 \times 2 = 26$$

Therefore,

$$\begin{aligned} P(3X + 2Y + 1 \geq 3) &= P(Z \geq 2) \\ &= P(\frac{Z-5}{\sqrt{26}} \geq \frac{2-5}{\sqrt{26}}) \\ &= Q(-\frac{3}{\sqrt{26}}) \end{aligned}$$

(b) Express $P(X^2 + Y^2 \geq 1 + 2XY)$ in terms of the Q function.
Solution:

$$\begin{aligned} P(X^2 + Y^2 \geq 1 + 2XY) &= P(X^2 + Y^2 - 2XY \geq 1) \\ &= P((X-Y)^2 \geq 1) \\ &= P(X - Y \geq 1) + P(X - Y \leq -1) \end{aligned}$$

Define $W = X - Y$, then W is a Gaussian random variable with

$$\mu_W = 1 \times 1 - 1 \times 1 = 0$$

and

$$\text{Var}(W) = \text{Var}(X) - 2\,\text{Cov}(X,Y) + \text{Var}(Y) = 2 + 2 \times 0 + 2 = 4$$

Therefore,

$$\begin{aligned} P(X^2 + Y^2 \geq 1 + 2XY) &= P(W \geq 1) + P(W \leq -1) \\ &= P(\frac{W}{2} \geq \frac{1}{2}) + P(\frac{W}{2} \leq -\frac{1}{2}) \\ &= Q(\frac{1}{2}) + \Phi(-\frac{1}{2}) \\ &= 2Q(\frac{1}{2}) \end{aligned}$$

9. **[Biased and unbiased estimators]**
Let $X_1, X_2, \cdots, X_n$ be independent Ber(p) random variables. Let $\hat{p}$ be the corresponding sample mean.

(a) Consider the estimator of the variance $\widehat{\sigma^2} = \hat{p}(1 - \hat{p})$. **True or False**: This estimator is unbiased. Justify your answer.
Solution: As $\sigma^2 = p(1-p)$, it is equivalent to determine if $E[\widehat{\sigma^2}] = p(1-p)$. As

$$\begin{aligned} E[\widehat{\sigma^2}] &= E[\hat{p}(1 - \hat{p})] \\ &= E\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\left(1 - \frac{1}{n}\sum_{i=1}^{n} X_i\right)\right] \\ &= E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] - E\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] \end{aligned}$$

with

$$E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i]$$
$$= \frac{1}{n}(np) = p$$

and (using $E[X_iX_j] = E[X_i]E[X_j]$ due to independence for all $i \neq j$)

$$\begin{aligned}
E\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] &= \frac{1}{n^2}E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n} E[X_i^2] + 2\sum_{i\neq j} E[X_iX_j]\right) \\
&= \frac{1}{n^2}\left(n(p^2 + p(1-p)) + 2\binom{n}{2}p^2\right) \\
&= \frac{1}{n^2}(np + n^2p^2 - np^2) \\
&= p^2 + \frac{p(1-p)}{n}
\end{aligned}$$

we find that

$$E[\widehat{\sigma^2}] = p(1-p) - \frac{p(1-p)}{n} \neq p(1-p)$$

Therefore, the estimator is biased, i.e. the statement is **false**.

(b) Let $\widehat{p^2} = \hat{p}^2 - \frac{\hat{p}(1-\hat{p})}{n-1}$ be an estimator of p^2. **True or False**: This estimator is unbiased. Justify your answer.

Solution: Similarly, we have to determine if $E[\widehat{p^2}] = p^2$. Using the results from (a),

$$\begin{aligned}
E[\widehat{p^2}] &= E[\hat{p}^2] - \frac{1}{n-1}E[\hat{p}(1-\hat{p})] \\
&= E\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] - \frac{1}{n-1}E[\hat{p}(1-\hat{p})] \\
&= p^2 + \frac{p(1-p)}{n} - \frac{1}{n-1}\left(p(1-p) - \frac{p(1-p)}{n}\right) \\
&= p^2 + p(1-p)\left(\frac{1}{n} - \frac{1}{n-1} + \frac{1}{n(n-1)}\right) \\
&= p^2
\end{aligned}$$

Therefore, the estimator is unbiased, i.e. the statement is **true**.

10. **[(Optional) Joint pmfs]**
Let X,Y be discrete, independent random variables with common pmf $p(u)$ for u in a discrete set. Define $W = \min\{X, Y\}$ and $Z = \max\{X, Y\}$. Find the joint pmf of W and Z.

Solution: Let A denote the discrete set. If $w, z \in A$ and $w < z$,

$$\begin{aligned} p_{WZ}(w,z) &= P(\min\{X,Y\} = w, \max\{X,Y\} = z) \\ &= P(X = w)P(Y = z) + P(X = z)P(Y = w) \\ &= p(w)p(z) + p(z)p(w) = 2p(w)p(z) \end{aligned}$$

Then, if $w, z \in A$ and $w = z$,

$$\begin{aligned} p_{WZ}(w,z) &= P(X = Y = w) \\ &= P(X = w)P(Y = w) \\ &= (p(w))^2 \end{aligned}$$

Otherwise (i.e. if $w, z \in A$ but $w > z$, or at least one of w and $z \notin A$), $p_{WZ}(w,z) = 0$.

To sum up,

$$p_{WZ}(w,z) = \begin{cases} 2p(w)p(z) & w, z \in A,\ w < z, \\ (p(w))^2 & w, z \in A,\ w = z, \\ 0 & \text{otherwise.} \end{cases}$$

11. **[(Optional) More on LLN]**
Let $X_1, X_2, \cdots$ be independent $N(0,1)$ random variables and $Y_1, Y_2, \cdots$ be independent $N(1,2)$ random variables. Suppose also that the two sequences are independent. Consider the function $h(x,y) = x^2 + y^2$ for $x, y \in \mathbb{R}$ and let $S_n = h(X_1, Y_1) + ... + h(X_n, Y_n)$. **True or False**: $\lim_{n\to\infty} P(|\frac{S_n}{n} - 4| > \frac{1}{2}) = 0$. Justify your answer.

Solution: Let $Z_i = h(X_i, Y_i)$ for $i = 1, 2, \ldots, n$, so $S_n = Z_1 + Z_2 + \cdots + Z_n$. Then

$$\mu_{Z_i} = E[X_i^2] + E[Y_i^2] = (0^2 + 1) + (1^2 + 2) = 4$$

Therefore, by Law of Large Numbers,

$$\lim_{n\to\infty} P(|\frac{S_n}{n} - 4| > \delta) = 0$$

for any $\delta > 0$. By plugging in $\delta = \frac{1}{2}$, we can conclude that the statement is **true**.

Extra Thinking: Find $\text{Var}(Z_i)$.

Reasoning:

$$\begin{aligned} \text{Var}(Z_i) &= \text{Var}(X_i^2) + \text{Var}(Y_i^2) \\ &= E[X_i^4] - (E[X_i^2])^2 + E[Y_i^4] - (E[Y_i^2])^2 \end{aligned}$$

Recall from Homework 9 that if $W_i \sim N(0,1)$, we have $E[W_i^4] = \frac{4!}{2^2 \times 2!} = 3$ and $E[W_i^m] = 0$ for odd m. Rewriting $Y_i = \sqrt{2}W_i + 1$, we have

$$\begin{aligned} E[Y_i^4] &= E[(\sqrt{2}W_i + 1)^4] \\ &= 4E[W_i^4] + 8\sqrt{2}E[W_i^3] + 12E[W_i^2] + 4\sqrt{2}E[W_i] + 1 \\ &= 4 \times 3 + 0 + 12 \times 1 + 0 + 1 \\ &= 25 \end{aligned}$$

Since $X_i \sim N(0,1)$ as well, $E[X_i^4] = 3$. Therefore,

$$\text{Var}(Z_i) = 3 - 1^2 + 25 - 3^2 = 18$$