University of Illinois **Spring 2024**

ECE 313: Problem Set 12

Due: Wednesday, May 1 at 7:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.5 - 4.11.

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. **Please write down your work and derivations. An answer without justification as of how it is found will not be accepted.** You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME AS IT APPEARS ON Canvas
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. [**Correlation and covariance**]
   Suppose $X \sim \text{Unif}(1, 2)$, and given $X = x$, Y is exponential with parameter $\lambda = x$.

   (a) Find the $\text{Cov}(X, Y)$.

   (b) Let $Z = X^2$. Find the marginal pdf and expected value of Z.

   (c) Find $\text{Cov}(X, Z)$.

2. [**Joint pdfs of functions of random variables**]
   Let X and Y have joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} e^{-u-v} & u, v \geq 0, \\ 0 & \text{else.} \end{cases}$$

   (a) Find the joint pdf of (W, Z), where $W = Y + 3X$ and $Z = Y^2$.

   (b) Find the pdf of $Z = XY$.

3. [**Order statistics**]
   Consider two random variables X and Y on the same probability space that are jointly continuous, with pdf $f_{X,Y}(x, y)$.

   (a) Find an expression for the pdfs of $Z = \max\{X, Y\}$ and $W = \min\{X, Y\}$. Rewrite the expression for the case that X and Y are independent, using their marginals.

   (b) Write down the expression for the pdf of the sum $Z + W$ using $f_{X,Y}(x, y)$.

(c) Assume now that X and Y are independent exponential random variables with the same parameter $\lambda = 1$. Find the pdfs of Z and W.

4. **[More on covariances]**
Consider two random variables X and Y on the same probability space.

(a) Assuming that X and Y have zero means and are uncorrelated, find the variances of X and Y given that $\mathrm{Var}(2X + 3Y) = 31$ and $\mathrm{Var}(X + Y) = 4$.

(b) Next, assume that you have no knowledge about whether X and Y are uncorrelated or not, and that the expectations of the variables are not necessarily zero. If $\mathrm{Var}(X+Y) = \mathrm{Var}(X - Y)$, are X and Y uncorrelated?

(c) If $\mathrm{Var}(X) = 2\,\mathrm{Var}(Y)$, do X and Y have to be uncorrelated? If so, justify your answer. If not, provide a counterexample.

5. **[Law of Large Numbers and Central Limit Theorem]**
A fair four-sided die is rolled n times. Let $S_n = X_1 + X_2 + \ldots + X_n$, where X_i is the number showing on the ith roll. Determine a condition on n such that the probability that the sample average $\frac{S_n}{n}$ is within 20% of the mean μ_X is greater than 0.92.

(a) Solve the problem using the form of the law of large numbers based on the Chebychev inequality.

(b) Solve the problem using the Gaussian approximation for S_n, which is suggested by the CLT. If you need to find $\mathcal{Q}(x)$ or $\Phi(x)$, round x to the nearest hundredth. (**Note**: Do not use the continuity correction for this question, because, unless $2.5n \pm (0.2)n\mu_X$ are integers, inserting the term 0.5 is not applicable).

6. **[Jointly Gaussian Random Variables I]**
Suppose X and Y are jointly Gaussian with $\mu_X = 2$, $\mu_Y = 1$, $\sigma_X^2 = 4$, $\sigma_Y^2 = 1$, and $\rho_{XY} = \frac{1}{16}$.

(a) Let $W = X + \alpha Y + \beta$.
Find the values of α and β that make X and W uncorrelated. Will these values α and β make X and W independent? Explain why.

(b) Let $Z = 3X + 2Y + 4$. Find the mean and variance of Z.

(c) Find $E[Y|Z = 11]$.

(d) Find $E[Y^2|Z = 11]$.

7. **[MMSE Estimation]**
Suppose X, Y have joint pmf

$$f_{XY}(x, y) = c(x + y), \qquad 0 \le x \le 2,\ 0 \le y \le 2$$

and 0 else.

(a) Find c.

(b) Find the unconstrained estimate of X given $Y = 1$.

(c) Find the best linear estimator of X given $Y = y_0$.

8. **[Jointly Gaussian Random Variables II]**
Let X, Y be independent $N(1, 2)$ random variables.

(a) Express $P(3X + 2Y + 1 \geq 3)$ in terms of the Q function.

(b) Express $P(X^2 + Y^2 \geq 1 + 2XY)$ in terms of the Q function.

9. **[Biased and unbiased estimators]**
Let $X_1, X_2, \cdots, X_n$ be independent Ber(p) random variables. Let $\hat{p}$ be the corresponding sample mean.

   (a) Consider the estimator of the variance $\widehat{\sigma^2} = \hat{p}(1 - \hat{p})$. **True or False**: This estimator is unbiased. Justify your answer.

   (b) Let $\widehat{p^2} = \hat{p}^2 - \frac{\hat{p}(1-\hat{p})}{n-1}$ be an estimator of p^2. **True or False**: This estimator is unbiased. Justify your answer.

10. **[(Optional) Joint pmfs]**
Let X,Y be discrete, independent random variables with common pmf $p(u)$ for u in a discrete set. Define $W = \min\{X, Y\}$ and $Z = \max\{X, Y\}$. Find the joint pmf of W and Z.

11. **[(Optional) More on LLN]**
Let $X_1, X_2, \cdots$ be independent $N(0, 1)$ random variables and $Y_1, Y_2, \cdots$ be independent $N(1, 2)$ random variables. Suppose also that the two sequences are independent. Consider the function $h(x, y) = x^2 + y^2$ for $x, y \in \mathbb{R}$ and let $S_n = h(X_1, Y_1) + ... + h(X_n, Y_n)$. **True or False**: $\lim_{n\to\infty} P(|\frac{S_n}{n} - 4| > \frac{1}{2}) = 0$. Justify your answer.