University of Illinois **Spring 2024**

ECE 313: Problem Set 11: Solutions

Due: Friday, April 19 at 7:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.8, 4.1 - 4.4

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. **Please write down your work and derivations. An answer without justification as of how it is found will not be accepted.** You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.
Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. [**Sampling Random Numbers from A Rayleigh Distribution**]

From first principles we can find that

$$E[X] = \frac{1}{\sigma^2}\int_0^\infty x^2 \exp(\frac{x^2}{2\sigma^2})dx$$

This integral can be solved by integration by parts. Let $u = x$ and $du = dx$. Let $dv = x\exp(-\frac{x^2}{2\sigma^2})dx$ and $v = -\sigma^2\exp(-\frac{x^2}{2\sigma^2})$

$$\frac{1}{\sigma^2}\int_0^\infty x^2 \exp(\frac{x^2}{2\sigma^2})dx = (-x\exp(-\frac{x^2}{2\sigma^2}))|_0^\infty + \int_0^\infty \exp(-\frac{x^2}{2\sigma^2})dx \quad (1)$$

$$= \int_0^\infty \exp(-\frac{x^2}{2\sigma^2})dx \quad (2)$$

$$= \frac{1}{2}\int_{-\infty}^\infty \exp(-\frac{x^2}{2\sigma^2})dx \quad (3)$$

$$= \frac{\sigma\sqrt{2\pi}}{2}\int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}}\exp(-\frac{x^2}{2\sigma^2})dx \quad (4)$$

This the term inside the integral becomes the PDF of a Gaussian integrated from negative infinity to infinity. We know this is equal to 1. So

$$E[X] = \sigma\sqrt{\frac{\pi}{2}}$$

The second moment is

$$E[X^2] = \frac{1}{\sigma^2}\int_0^\infty x^3 \exp(\frac{x^2}{2\sigma^2})dx$$

which can again be solved by integration by parts. Let $u = x^2$ and $du = 2xdx$. Let $dv = x\exp(-\frac{x^2}{2\sigma^2})dx$ and $v = -\sigma^2\exp(-\frac{x^2}{2\sigma^2})$

$$\frac{1}{\sigma^2}\int_0^\infty x^3 \exp(\frac{x^2}{2\sigma^2})dx = (-x^2\exp(-\frac{x^2}{2\sigma^2}))|_0^\infty + 2\sigma^2\int_0^\infty \frac{x}{\sigma^2}\exp(-\frac{x^2}{2\sigma^2})dx \tag{5}$$

$$= 2\sigma^2 \tag{6}$$

This is because the integral on the right can be realized as the PDF of the Rayleigh distribution integrated across its bounds. So,

$$Var(X) = 2\sigma^2 - \sigma^2\frac{\pi}{2} = \sigma^2\frac{4-\pi}{2}$$

One way to establish equivalent distributions is to equate the CDFs of the two distributions. By integration by parts, we can find the CDF of the Rayleigh distribution to be $1-\exp(\frac{-c^2}{2\sigma^2})$ for $c \geq 0$. Following section 3.8.2 in the lecture notes, we can find that $g(U) = F^{-1}(U)$. Inverting the CDF we found gives

$$g(U) = \sigma\sqrt{-2\ln(1-U)}$$

2. **[Joint PDFs]**

$$\int_0^1\int_0^1 f_{XY}(x,y)dydx = \int_0^1(\frac{c}{3}+x)dx$$
$$= \frac{c}{3}+\frac{1}{2}$$
$$= 1$$

so $c = \frac{3}{2}$.

The marginal pdf $f_X(x)$ is given by

$$f_X(x) = \int_0^1 f_{XY}(x,y)dy$$
$$= x + \frac{1}{2}$$

for $0 \leq x \leq 1$ and 0 otherwise.

Similarly, the marginal pdf $f_Y(y)$ is given by

$$f_Y(y) = \int_0^1 f_{XY}(x,y)dx$$
$$= \frac{3y^2}{2}+\frac{1}{2}$$

for $0 \leq y \leq 1$ and 0 otherwise.

Finally,

$$
\begin{aligned}
P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2}) &= \int_0^{1/2} \int_0^{1/2} f_{XY}(x,y) dy dx \\
&= \int_0^{1/2} (\frac{1}{16} + \frac{x}{2}) dx \\
&= \frac{1}{32} + \frac{1}{16} = \frac{3}{32}
\end{aligned}
$$

3. **[Joint CDFs]**

(a)

$$
\begin{aligned}
\int_0^1 \int_0^x f_{XY}(x,y) dy dx &= \int_0^1 \frac{cx^4}{2} dx \\
&= \frac{c}{10} \\
&= 1
\end{aligned}
$$

so $c = 10$.

(b) The marginal pdf $f_X(x)$ is given by

$$
\begin{aligned}
f_X(x) &= \int_0^x f_{XY}(x,y) dy \\
&= 5x^4
\end{aligned}
$$

for $0 \leq x \leq 1$ and 0 otherwise.

Similarly, the marginal pdf $f_Y(y)$ is given by

$$
\begin{aligned}
f_Y(y) &= \int_y^1 f_{XY}(x,y) dx \\
&= \frac{10}{3}(y - y^4)
\end{aligned}
$$

for $0 \leq y \leq 1$ and 0 otherwise.

(c)

$$
\begin{aligned}
P(Y \leq \frac{X}{2}) &= \int_0^1 \int_0^{x/2} f_{XY}(x,y) dy dx \\
&= \int_0^1 \frac{5x^4}{4} dx \\
&= \frac{1}{4}
\end{aligned}
$$

(d)

$$
\begin{aligned}
P(Y \leq \frac{X}{4} | Y \leq \frac{X}{2}) &= \frac{P(Y \leq \frac{X}{4})}{P(Y \leq \frac{X}{2})} \\
&= 4\int_0^1 \int_0^{x/4} f_{XY}(x,y)dydx \\
&= 4\int_0^1 \frac{5x^4}{16}dx \\
&= 4 \times \frac{1}{16} = \frac{1}{4}
\end{aligned}
$$

4. **[Independent or not?]**

(a) Yes, X and Y are independent because $f_{X,Y}(u,v) = f_X(u)f_Y(v)$, where $f_X(u) = \sqrt{C}e^{-u^3}$ and $f_Y \equiv f_X$.

(b) No, X and Y are not independent. The support of the pdf is not a product set. For example, consider the following points $f_{X,Y}(0.1, 0.6) > 0$ and $f_{X,Y}(0.6, 0.1) > 0$ while $f_{X,Y}(0.6, 0.6) = 0$.

(c) No, X and Y are not independent. Consider the events $\{X < 0.5\}$ and $\{Y > 1.5\}$. Clearly, $P\{X < 0.5, Y > 1.5\} = 0$. However, $P\{X < 0.5\} > 0$ and $P\{Y > 1.5\} > 0$, which would imply $P\{X < 0.5, Y > 1.5\} \neq P\{X < 0.5\}P\{Y > 1.5\}$.

Alternatively, you can show that the support is not a product set, e.g. $f_{X,Y}(0.7, 0.8) > 0$ and $f_{X,Y}(0.9, 1) > 0$ while $f_{X,Y}(0.9, 0.8) = 0$.

(d) Yes, X and Y are independent. For a fixed $X = u$,

$$Z + X = Z + u \sim \text{Uniform}(u, 1 + u)$$

Then, the fractional part of $Z + u \sim \text{Uniform}(0, 1)$. Hence, $P(Y \leq v | X = u) = v$, which does not depend on X.

Alternatively, this can be found with a bit of math. Since the support of Y is $[0, 1]$, let's assume that $v \in [0, 1]$.

$$
\begin{aligned}
P(Y \leq v | X = u) &= P(X + Z - \lfloor X + Z \rfloor \leq v | X = u) && (7) \\
&= P(u + Z - \lfloor u + Z \rfloor \leq v) && (8) \\
& && (9)
\end{aligned}
$$

Note that $\lfloor u + Z \rfloor = 0$ if $u + Z < 1$ and 1 if $u + Z > 1$. So we can use total probability law to break this problem into more manageable pieces.

$$
\begin{aligned}
P(u + Z - \lfloor u + Z \rfloor \leq v) &= P(u + Z \leq v, Z < 1 - u) + P(u + Z - 1 \leq v, Z > 1 - u) && (10) \\
&= P(Z \leq v - u) + P(Z \leq 1 + v - u, Z > 1 - u) && (11)
\end{aligned}
$$

Now, if $v \geq u$,

$$
\begin{aligned}
P(Z \leq v - u) + P(Z \leq 1 + v - u, Z > 1 - u) &= P(Z \leq v - u) + P(Z > 1 - u) && (12) \\
&= v - u + 1 - (1 - u) && (13) \\
&= v && (14)
\end{aligned}
$$

Now, if $v < u$,

$$P(Z \leq v - u) + P(Z \leq 1 + v - u, Z > 1 - u) = 0 + P(1 - u < Z \leq 1 + v - u) \tag{15}$$
$$= 1 + v - u - (1 - u) \tag{16}$$
$$= v \tag{17}$$

So $P(Y \leq v | X = u) = v$, which does not depend on u.