University of Illinois **Spring 2024**

ECE 313: Problem Set 11

Due: Friday, April 19 at 7:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.8, 4.1 - 4.4

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. **Please write down your work and derivations. An answer without justification as of how it is found will not be accepted.** You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. [**Sampling Random Numbers from A Rayleigh Distribution**]
   A random variable X is said to have a Rayleigh distribution with parameter $\sigma > 0$ (frequently encountered when modelling wireless communication systems) if its pdf is of the form

   $$f_X(x) = \frac{x}{\sigma^2}\ \exp\{\frac{-x^2}{2\sigma^2}\},\ x \geq 0,$$

   and zero otherwise. Find the mean and variance of the random variable, as well as the function $g(U)$ that you need to apply to a uniform in $[0, 1]$ random variable U in order to generate a Rayleigh random variable.

2. [**Joint PDFs**]
   Let X and Y be two jointly continuous random variables with joint PDF

   $$f_{XY}(x, y) = \begin{cases} x + cy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   Find the constant c. Find the marginals of X and Y, as well as $P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2})$.

3. [**Joint CDFs**]
   Let X and Y be two jointly continuous random variables with joint PDF

   $$f_{XY}(x, y) = \begin{cases} cx^2 y & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the constant c.

(b) Find the marginal pdfs, $f_X(x)$ and $f_Y(y)$.

(c) Find $P(Y \leq \frac{X}{2})$.

(d) Find $P(Y \leq \frac{X}{4} | Y \leq \frac{X}{2})$.

4. **[Independent or not?]**
Decide whether X and Y are independent for each of the following cases. The constant C in each case represents the value making the pdf integrate to one. Justify your answer.

(a)
$$f_{X,Y}(u,v) = \begin{cases} Ce^{-(u^3+v^3)}, & \text{if } 0 \leq u \leq 1 \text{ and } 0 \leq v \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

(b)
$$f_{X,Y}(u,v) = \begin{cases} 2, & \text{if } u \geq 0 \text{ and } 0 \leq v \leq 1-u; \\ 0, & \text{otherwise.} \end{cases}$$

(c) Let X and Z be independent and uniformly distributed in $[0,1]$ and $Y = X + Z$.

(d) Let X and Z be independent and uniformly distributed $(0,1)$ and $Y = X + Z - \lfloor X + Z \rfloor$ (Y is the fractional part of $X+Z$. For example, if $X+Z = 1.21$, then $Y = 1.21 - \lfloor 1.21 \rfloor = 1.21 - 1 = 0.21$).
Hint: Compute $P(Y \leq v | X = u)$.