University of Illinois **Spring 2024**

ECE 313: Problem Set 10: Solutions

Due: Friday, April 5 at 7 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.7, 3.8.1, 3.8.2, and 3.10

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. **Please write down your work and derivations. An answer without justification as of how it is found will not be accepted.** You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.
Please write on the top right corner of the first page:
NAME
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SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Maximum Likelihood Parameter Estimation]**

   (a) From the scaling rule, the likelihood function $f_\theta(v) = f_Y(v)$ with $\theta = a$ is given by:

$$f_\theta(v) = \frac{1}{|\theta|} f_X\left(\frac{v-b}{\theta}\right) = \frac{1}{|\theta|}\mathbb{I}_{\{|\frac{v-1}{\theta}-\frac{1}{2}|\le\frac{1}{2}\}}$$
$$f_\theta(3) = \frac{1}{|\theta|}\mathbb{I}_{\{|\frac{2}{\theta}-\frac{1}{2}|\le\frac{1}{2}\}}$$

   Where $\mathbb{I}$ is the indicator function–it evaluates to 1 when its condition is true, and 0 otherwise. The likelihood function $f_\theta(3)$ is maximized when $|\theta|$ is minimized while ensuring the indicator function evaluates to 1. Therefore, θ needs to satisfy:

$$\left|\frac{2}{\theta} - \frac{1}{2}\right| \le \frac{1}{2} \implies 0 \le 2 \le a$$

   Hence, $\hat{a}_{\text{ML}} = 2$. Since $Y = \hat{a}_{\text{ML}}X + 1$, $Y \sim [1, 3]$ and

$$f_Y(v) = \frac{1}{2}\mathbb{I}_{\{|u-2|\le 1\}}$$

   Alternatively, you can write this in piecewise form

$$f_Y(v) = \begin{cases} \frac{1}{2} & 1 < v \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Alternative Approach: Since we have observed $Y = 3$ and we want to maximize $f_Y(3)$, we need $f_Y(3) > 0$, which indicates that $aX + 1 = 3$ is possible. As the support of X is $[0, 1]$, $a \geq 2$. With $f_Y(v) = \frac{1}{|a|} \cdot f_X(\frac{v-b}{a})$, we know $f_Y(3) = \frac{1}{|a|}$ under the condition $a \geq 2$, and thus $\hat{a}_{\mathrm{ML}} = 2$ since it maximizes $\frac{1}{|a|}$. We can finally get $f_Y(v)$ by plugging in $a = \hat{a}_{\mathrm{ML}}$.

(b) The likelihood function $f_\theta(u) = f_X(u)$ with $\theta = a$. From the linear scaling rule, we get

$$f_X(\frac{v-1}{a}) = |a| f_Y(v) \implies f_\theta(\frac{v-1}{\theta}) = |\theta| f_Y(v)$$

Substituting, $u = \frac{v-1}{\theta}$, we obtain,

$$f_\theta(u) = |\theta| f_Y(\theta u + 1) \implies f_\theta(3) = |\theta| f_Y(3\theta + 1) = |\theta| \mathbb{I}_{\{|3\theta+1-0.5| \leq 0.5\}}$$

The likelihood function $f_\theta(3)$ is maximized when $|\theta|$ achieves the highest value while keeping the indicator function equal to unity. This implies,

$$|3\theta + 0.5| \leq 0.5 \implies -\frac{1}{3} \leq \theta \leq 0 \implies \hat{a}_{\mathrm{ML}} = -\frac{1}{3}$$

Hence, $\hat{a}_{\mathrm{ML}} = -\frac{1}{3}$, and since $X = \frac{Y-1}{\hat{a}_{\mathrm{ML}}}$, $X \sim [0, 3]$ and

$$f_X(u) = \frac{1}{3} \mathbb{I}_{\{|u-1.5| \leq 1.5\}}$$

Alternatively, you can write this in piecewise form

$$f_X(v) = \begin{cases} \frac{1}{3} & 0 < v \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Alternative Approach: Since $X = \frac{1}{a}Y - \frac{1}{a}$, we have observed $X = 3$, and we want to maximize $f_X(3)$, we need $f_X(3) > 0$, which indicates that $\frac{1}{a}Y - \frac{1}{a} = 3$ is possible. As the support of Y is $[0, 1]$, $-\frac{1}{3} \leq a \leq 0$. With $f_X(u) = |a| \cdot f_Y(\frac{v+1/a}{1/a})$, we know $f_X(3) = |a|$ under the condition $-\frac{1}{3} \leq a \leq 0$, and thus $\hat{a}_{\mathrm{ML}} = -\frac{1}{3}$ since it maximizes $|a|$. We can finally get $f_X(u)$ by plugging in $a = \hat{a}_{ML}$.

2. **[Function of a RV]**

First, we sketch $v = g(u)$ and $f_X(u)$ to find that Y is continuous-type RV with support $[-a, a]$.

Hence,

$$F_Y(c) = \begin{cases} 0, & \text{for} \quad c \leq -a \\ 1, & \text{for} \quad c \geq a \end{cases}$$

For $u \in [0, \pi]$, $\cos(u)$ is monotonically decreasing. Thus, for $-a < c < a$:

$$\begin{aligned} F_Y(c) &= P\{a\cos(X) \leq c\} = P\{X \geq \cos^{-1}(\frac{c}{a})\} \\ &= \frac{1}{\pi} \int_{\cos^{-1}(\frac{c}{a})}^{\pi} du = 1 - \frac{\cos^{-1}(\frac{c}{a})}{\pi} \end{aligned}$$

Differentiating $F_Y(c)$ w.r.t. c, and using the relationship $\frac{d\cos^{-1}(y)}{dy} = -\frac{1}{\sqrt{1-y^2}}$, we get the following pdf:

$$f_Y(c) = \frac{dF_Y(c)}{dc} = \frac{1}{\sqrt{1-(\frac{c}{a})^2}} \times \frac{1}{a\pi} = \frac{1}{\pi}\frac{1}{\sqrt{a^2-c^2}}, \qquad \text{for} \quad -a < c < a$$
$$f_Y(c) = 0, \qquad \text{for} \quad |c| \geq a$$

3. **[Function of a RV II]**

(a) If $-5 \leq u < 0$, $h(X) \in (0, 2.5]$
If $0 \leq u < 2$, $h(X) \in (0, 4]$
If $2 \leq u \leq 5$, $h(X) \in [4, 25]$
Y can be realized anywhere in the union of these three intervals, so the support of Y is $(0, 25]$.
Because the number of real numbers in the interval $(0, 25]$ is uncountably infinite, Y is a continuous-type random variable.

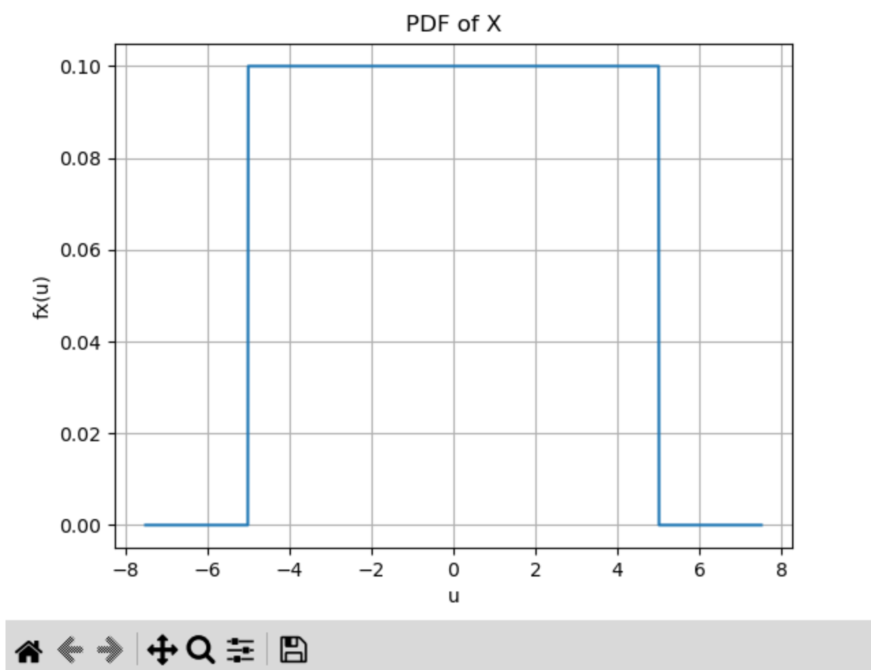

Figure 1: Problem 3 Probability Density Function of X

(b) If $u \leq 0$, then $F_Y(u) = 0$. Similarly, if $u > 25$, then $F_Y(u) = 1$. Now we have to consider the values in between.
We should note that $h(X)$ is not one-to-one. For example, $h(X) = 1$ can be because $X = -2$ or $X = 0.5$.

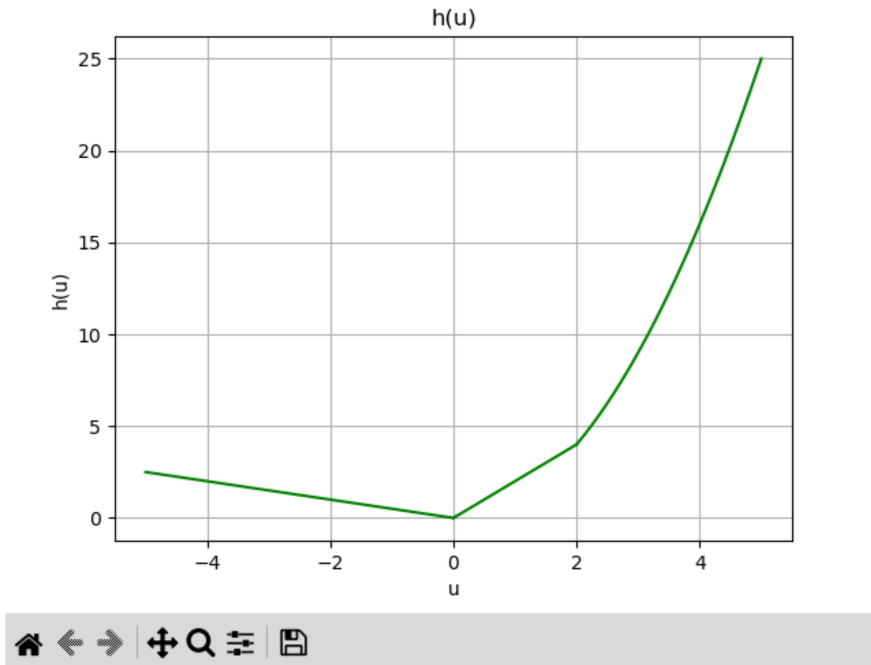

Figure 2: Problem 3 Plot of Function $h(u)$

If $0 < u \leq 2.5$

$$
\begin{aligned}
P(Y < u) &= P(h(X) < u) && (1)\\
&= P(X \in [-2u, 0) \cup X \in [0, \frac{u}{2})) && (2)\\
&= P(-2u < X \leq \frac{u}{2}) && (3)\\
&= \frac{u}{4} && (4)
\end{aligned}
$$

If $2.5 < u \leq 4$

$$
\begin{aligned}
P(Y < u) &= P(h(X) < u) && (5)\\
&= P(X \in [-5, 0) \cup X \in [0, \frac{u}{2})) && (6)\\
&= P(-5 < X \leq \frac{u}{2}) && (7)\\
&= \frac{u + 10}{20} && (8)
\end{aligned}
$$

If $4 < u \leq 25$.

$$
\begin{aligned}
P(Y < u) &= P(h(X) < u) && (9)\\
&= P(X \in [-5, 0) \cup X \in [0, 2) \cup X \in [2, \sqrt{u})) && (10)\\
&= P(-5 < X \leq \sqrt{u}) && (11)\\
&= \frac{\sqrt{u} + 5}{10} && (12)
\end{aligned}
$$

Therefore the CDF of Y is

$$F_Y(u) = \begin{cases} 0 & u \leq 0 \\ \frac{u}{4} & 0 < u \leq 2.5 \\ \frac{u+10}{20} & 2.5 < u \leq 4 \\ \frac{\sqrt{u}+5}{10} & 4 < u \leq 25 \\ 1 & u > 25 \end{cases}$$

(c) The PDF of Y is the derivative of the CDF.

$$f_Y(u) = \begin{cases} \frac{1}{4} & 0 < u \leq 2.5 \\ \frac{1}{20} & 2.5 < u \leq 4 \\ \frac{1}{20\sqrt{u}} & 4 < u \leq 25 \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y] = \int_{-\infty}^{\infty} u f_Y(u) du \tag{13}$$

$$= \int_0^{2.5} u\frac{1}{4}du + \int_{2.5}^{4} u\frac{1}{20}du + \int_4^{25} \sqrt{u}\frac{1}{20}du \tag{14}$$

$$= \int_0^{2.5} \frac{u}{4}du + \int_{2.5}^{4} u\frac{1}{20}du + \int_4^{25} u\frac{1}{20\sqrt{u}}du \tag{15}$$

$$= 4.925 \tag{16}$$

4. **[Function of a RV III]**

(a) The pdf of X is

$$f_X(u) = \begin{cases} \frac{1}{10} & 0 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

With the definition of $h(u)$, there is

$$h(u) = \begin{cases} 0 & u = 0 \\ 1 & 0 < u \leq 1 \\ 2 & 1 < u \leq 2 \\ 3 & 2 < u \leq 3 \\ 4 & 3 < u \leq 4 \\ 5 & 4 < u \leq 5 \\ 6 & 5 < u \leq 6 \\ 0 & 6 < u \leq 7 \\ 1 & 7 < u \leq 8 \\ 2 & 8 < u \leq 9 \\ 3 & 9 < u \leq 10 \end{cases}$$

The support of Z is then all possible values of $h(u)$, i.e. $\{0, 1, 2, 3, 4, 5, 6\}$. Since the support of Z is discrete (there is a finite number of realizations for the random variable Z), it is a discrete-type random variable.

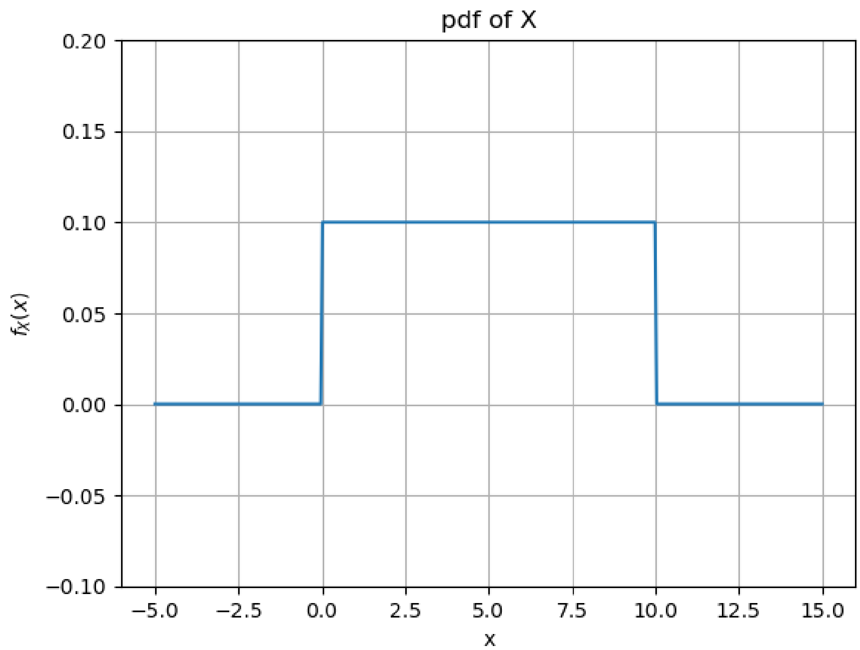

Figure 3: Problem 4 Probability Density Function of X

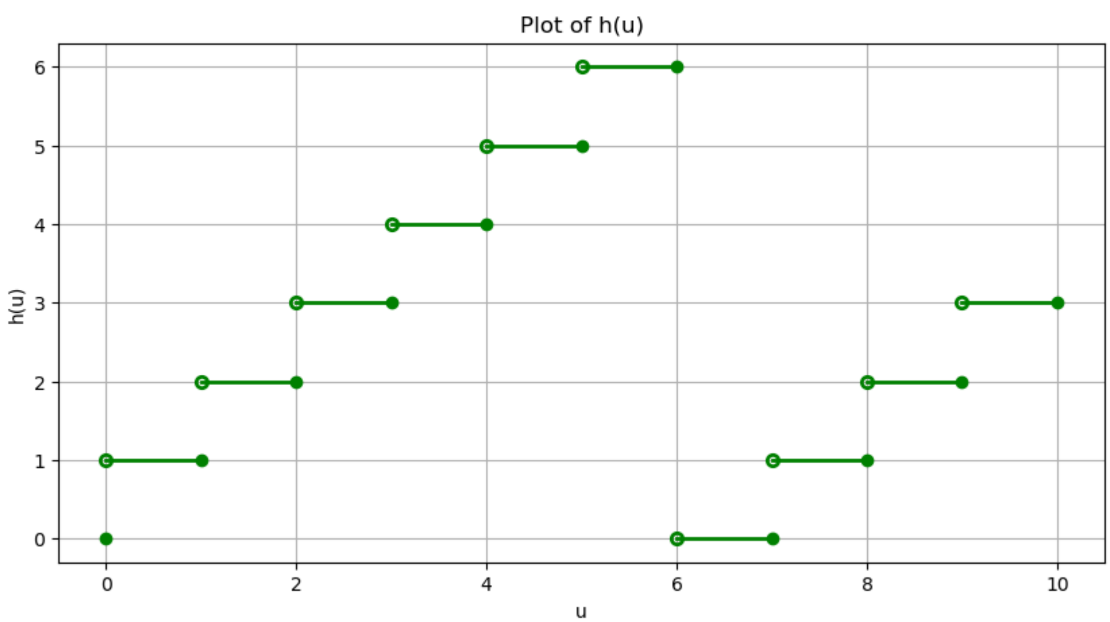

Figure 4: Problem 4 Plot of Function $h(u)$

(b) According to $h(u)$, when $v < u \leq v+1$ for $v \in \{0, 1, \ldots, 9\}$, $\lceil u \rceil \text{mod}(7) = (v+1) \text{ mod}(7)$, then as $P(v < u \leq v+1) = \frac{1}{10}$, we may write the pmf of Z as

$$p_Z(w) = \begin{cases} \frac{1}{10} & w = 0, 4, 5, 6 \\ \frac{1}{5} & w = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

Note that although $X = 0$ also gives $Z = 0$, we have $P(X = 0) = 0$ since X is a continuous-type random variable, so $p_Z(w)$ is not affected.