University of Illinois **Spring 2024**

ECE 313: Problem Set 9: Solutions

Due: Friday, March 29 at 7 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.6.1 - 3.6.3

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. **Please write down your work and derivations. An answer without justification as of how it is found will not be accepted.** You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Moments and Approximations]**

   (a) By LOTUS,

   $$E[X^{2k+1}] = \int_{-\infty}^{\infty} u^{2k+1} f_X(u)du = 0$$

   because the integrand is an odd function.

   (b) Applying the trick of integration by parts, we have

   $$\begin{aligned}
   E[X^{2k}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^{2k} e^{-u^2/2} du \\
   &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^{2k-1} \cdot u e^{-u^2/2} du \\
   &= \frac{1}{\sqrt{2\pi}} \left(\left[-u^{2k-1} e^{-u^2/2} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (2k-1) u^{2k-2} e^{-u^2/2} du \right) \\
   &= (2k-1) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^{2k-2} e^{-u^2/2} du \\
   &= (2k-1) \cdot E[X^{2k-2}]
   \end{aligned}$$

This recursive formula implies that for $k \geq 1$:

$$\begin{aligned} E[X^{2k}] &= (2k-1)(2k-3)\cdots \times 3 \times 1 \times E[X^0] \\ &= (2k-1)(2k-3)\cdots \times 3 \times 1 \\ &= \frac{(2k)!}{(2k)(2k-2)\cdots \times 4 \times 2} \\ &= \frac{(2k)!}{2^k k!} \end{aligned}$$

This result also holds for $k = 0$ as $E[X^0] = 1 = \frac{(2\times 0)!}{2^0 \times 0!}$. To sum up,

$$E[X^{2k}] = \frac{(2k)!}{2^k k!}, \quad k = 0, 1, 2, \ldots$$

(c) The Gaussian random variable $\tilde{Z}$ for approximation should have $\mu_{\tilde{Z}} = np = \frac{11}{5}$ and $\sigma^2_{\tilde{Z}} = np(1-p) = \frac{44}{25}$.

$$\begin{aligned} P(Z = 5) &= P(4.5 \leq \tilde{Z} \leq 5.5) \\ &= P(\frac{5 \times (4.5 - 11/5)}{2\sqrt{11}} \leq X \leq \frac{5 \times (5.5 - 11/5)}{2\sqrt{11}}) \\ &= \Phi(\frac{33}{4\sqrt{11}}) - \Phi(\frac{23}{4\sqrt{11}}) \\ &\approx \Phi(2.49) - \Phi(1.73) \end{aligned}$$

2. **[Gaussian Probabilities]**

(a) Let $Z \sim N(0, 1)$, i.e. the standard Gaussian distribution (and same for other questions).

$$\begin{aligned} P(|X-2| - 1 < 0) &= P(1 < X < 3) \\ &= P(\frac{1+1}{5} < Z < \frac{3+1}{5}) \\ &= \Phi(0.8) - \Phi(0.4) \\ &\approx 0.1327 \end{aligned}$$

(b) As $|m|$ increases,

$$P(X^2 \leq m^2) = P(-|m| \leq X \leq |m|) = \Phi(0.2 + 0.2|m|) - \Phi(0.2 - 0.2|m|)$$

also increases. Note that

- when $|m| = 2$, $P(X^2 \leq 4) = \Phi(0.6) - \Phi(-0.2) = \Phi(0.6) - Q(0.2) \approx 0.305$;
- when $|m| = 3$, $P(X^2 \leq 9) = \Phi(0.8) - \Phi(-0.4) = \Phi(0.8) - Q(0.4) \approx 0.4435$.

Therefore, $|m| \leq 2$, i.e. all possible values for m are $\{-2, -1, 0, 1, 2\}$.

(c) and 0 for $v < 0$. Find the value of the constant K.
We know that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{v^2}{2}) dv = 1$$

As the integrand is an even function, we further have

$$\int_0^\infty \frac{1}{\sqrt{2\pi}} \exp(-\frac{v^2}{2})dv = \frac{1}{2}$$

and thus

$$\int_0^\infty \frac{2}{\sqrt{2\pi}} \exp(-\frac{v^2}{2})dv = 1 = \int_0^\infty f(v)dv$$

Therefore, $K = 2$.

3. **[Scaling Rule for pdfs]**

(a)

$$E[X] = \int_{-0.5}^{0} udu + \int_0^1 (u - u^2)du = \frac{1}{24}$$
$$E[X^2] = \int_{-0.5}^{0} u^2du + \int_0^1 (u^2 - u^3)du = \frac{1}{8}$$
$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{71}{576}$$

Then we have

$$E[Y] = 2E[X] - 3 = -\frac{35}{12}$$
$$\text{Var}(Y) = 2^2 \cdot \text{Var}(X) = \frac{71}{144}$$

(b)

$$E[Z] = -E[X] + 2 = \frac{47}{24}$$
$$\text{Var}(Z) = (-1)^2 \cdot \text{Var}(X) = \frac{71}{576}$$

(c) Using the scaling rule for pdfs (Eq.(3.8) in course notes),

$$\begin{aligned} f_Y(v) &= \frac{1}{|2|} \cdot f_X(\frac{v+3}{2}) \\ &= \begin{cases} \frac{1}{2} & -4 \le v \le -3, \\ -\frac{v+1}{4} & -3 \le v \le -1. \end{cases} \end{aligned}$$

$$\begin{aligned} f_Z(w) &= \frac{1}{|-1|} \cdot f_X(\frac{v-2}{-1}) \\ &= \begin{cases} 1 & 2 \le w \le 2.5, \\ w-1 & 1 \le w \le 2. \end{cases} \end{aligned}$$

The plots are shown in Fig.1 and Fig.2.

4. **[The Table of Q-Function]**

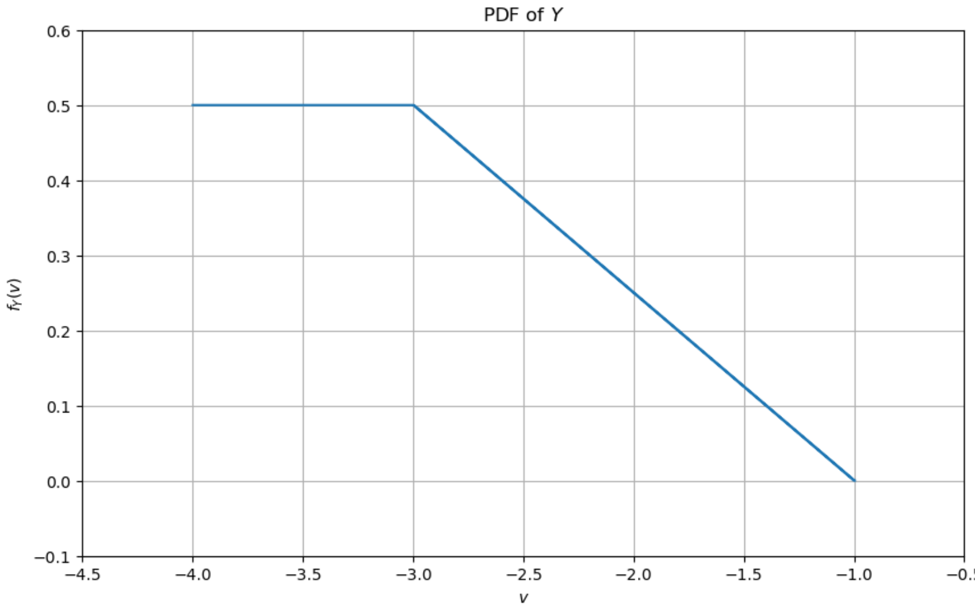

Figure 1: PDF of Random Variable Y

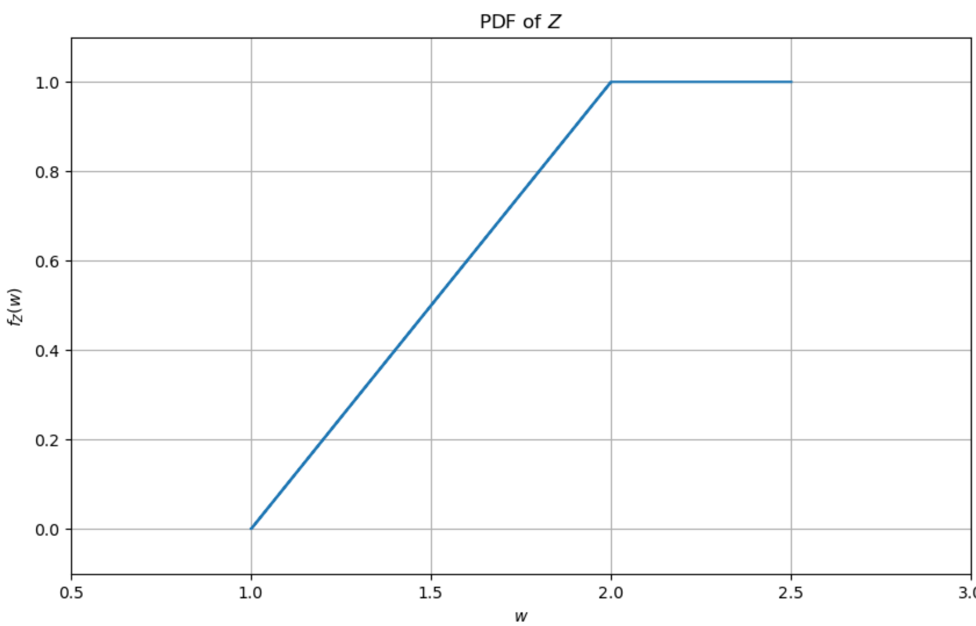

Figure 2: PDF of Random Variable Z

(a)

$$P(X \leq 6.20) = P(\frac{X-\mu}{\sigma_X} \leq \frac{6.20-\mu}{\sigma_X}) = 0.9032$$

Similarly,

$$P(X \geq 2) = P(\frac{X-\mu}{\sigma_X} \leq \frac{2-\mu}{\sigma_X}) = 0.40129$$

By using the table of Q-functions (found in the appendix of the lecture notes), we can find that

$$\frac{6.20-\mu}{\sigma_X} = 1.30$$
$$\frac{2-\mu}{\sigma_X} = 0.25$$

By jointly solving these 2 equations, we obtain $\mu = 1$ and $\sigma_X = 4$.

(b) Since $P(X \geq 4) = P(X \leq 0)$, we have $P(\frac{X-\mu}{\sigma_X} \geq \frac{4-\mu}{\sigma}) = P(\frac{X-\mu}{\sigma_X} \leq \frac{-\mu}{\sigma})$ which implies $Q(\frac{4-\mu}{\sigma}) = \Phi(\frac{-\mu}{\sigma})$. By using the relation $Q(x) = \Phi(-x)$ we obtain $\frac{4-\mu}{\sigma} = \frac{\mu}{\sigma}$. Thus, we have $\mu = 2$.

Then, we use $P(X \geq 1) = P(\frac{X-2}{\sigma_X} \geq \frac{-1}{\sigma_X}) = 0.69146$. By using the table of Q-functions, we obtain $\frac{-1}{\sigma_X} = -0.5$ which gives $\sigma_X = 2$.

(c) If we know the first and second moments of X, we can find the mean and variance.

$$E[(X+2)(X+1)] = E[X^2+3X+2] \tag{1}$$
$$= E[X^2] + 3E[X] + 2 \tag{2}$$

So $E[X^2] + 3E[X] = 5$ Similarly,

$$E[X(X+1)] = E[X^2+X] \tag{3}$$
$$= E[X^2] + E[X] \tag{4}$$

So $E[X^2] + E[X] = 5$ Solving this system of equations shows that $E[X] = 0$ and $E[X^2] = 5$. As a result we can conclude that $\mu = 0$ and $\sigma = \sqrt{5-0} = \sqrt{5}$

5. **[Communication in Gaussian Noise]**

(a)

$$E[Y] = E[2+Z] \tag{5}$$
$$= 2 + E[Z] \tag{6}$$
$$= 2 \tag{7}$$

$$Var(Y) = E[(Y - E[Y])^2] \tag{8}$$
$$= E[(Z+2-2)^2] \tag{9}$$
$$= E[Z^2] \tag{10}$$
$$= Var(Z) + E[Z] \tag{11}$$
$$= 4 \tag{12}$$

Intuitively, this should make sense, as "shifting" a random variable should affect the mean and not the variance. The received signal Y follows a Gaussian distribution with $\mu_Y = 2$ and $\sigma_Y^2 = 4$. The pdf is

$$P(Y = y) = \frac{1}{2\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(\frac{y-2}{2})^2\right\}$$

(b) Let Y_2 and Y_{-2} be the received signal when $x = 2$ and $x = -2$ are transmitted respectively. Their distributions are

$$Y_2 \sim N(2, 4)$$
$$Y_{-2} \sim N(-2, 4)$$

The individual probabilities of error for the two symbols transmitted can be written as $P(Y \leq 0|x = 2)$ and $P(Y > 0|x = -2)$.

$$\begin{aligned} P(Y \leq 0|x = 2) &= P(Y_2 < 0) \\ &= P(\frac{Y_2 - 2}{2} < -1) \\ &= \Phi(-1) = Q(1) \end{aligned}$$

$$\begin{aligned} P(Y > 0|x = -2) &= P(Y_{-2} > 0) \\ &= P(\frac{Y_{-2} + 2}{2} > 1) \\ &= Q(1) \end{aligned}$$

The receiver's probability of error is

$$\begin{aligned} p_{error} &= P(x = 2)P(Y \leq 0|x = 2) + P(x = -2)P(Y > 0|x = -2) \\ &= Q(1)(\frac{1}{2} + \frac{1}{2}) \\ &= Q(1) \end{aligned}$$

(c) Similar to the previous part, we define Y_{-4}, Y_0 and Y_4 as

$$Y_4 \sim N(4, 4)$$
$$Y_0 \sim N(0, 4)$$
$$Y_{-4} \sim N(-4, 4)$$

The individual probabilities of error are

$$\begin{aligned} P(Y \leq 2|x = 4) &= P(Y_4 \leq 2) \\ &= P(\frac{Y_4 - 4}{2} \leq -1) \\ &= \Phi(-1) = Q(1) \end{aligned}$$

$$\begin{aligned} P(Y > 2 \cup Y < -2|x = 0) &= P(Y_0 > 2 \cup Y_0 < -2) \\ &= P(\frac{Y_0}{2} > 1) + P(\frac{Y_0}{2} < -1) \\ &= Q(1) + \Phi(-1) = 2Q(1) \end{aligned}$$

$$\begin{aligned} P(Y \geq -2|x=-4) &= P(Y_{-4} \geq -2) \\ &= P(\frac{Y_{-4}+4}{2} \geq 1) \\ &= Q(1) \end{aligned}$$

The receiver's probability of error is

$$\begin{aligned} p_{error} = &P(x=4)P(Y \leq 2|x=4) + \\ &P(x=0)P(Y > 2 \cup Y < -2|x=0) + \\ &P(x=-4)P(Y \geq -2|x=-4) \\ &= Q(1)(\frac{1}{3} + \frac{2}{3} + \frac{1}{3}) \\ &= \frac{4}{3}Q(1) \end{aligned}$$