ECE 313: Problem Set 9

Due: Friday, March 29 at 7 p.m.

Reading: ECE 313 Course Notes, Sections 3.6.1 - 3.6.3

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. Please write down your work and derivations. An answer without justification as of how it is found will not be accepted. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:

NAME NETID SECTION PROBLEM SET

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. 5 points will be deducted for incorrectly assigned page numbers.

1. [Moments and Approximations]

Let X be a standard Gaussian random variable with PDF $f_X(u)$.

- (a) Find the moments of X of odd order i.e. find $E[X^{2k+1}]$ for k = 0, 1, 2, ...
- (b) Show that

$$E[X^{2k}] = \frac{(2k)!}{2^k k!}$$

(**Hint**: Start with $E[X^{2k}] = \int_{-\infty}^{\infty} u^{2k} f_X(u) du$, write $u^{2k} = u^{2k-1}u$ and use integration by parts.)

(c) This part is separate from parts (a) and (b). Let Z be a binomial random variable with parameters n = 11 and p = 0.2. Find the normal approximation to P(Z = 5) using continuity correction. Your answer should be in terms of the standard normal CDF Φ or the Q-function.

2. [Gaussian Probabilities]

Suppose we have two random variables $X \sim N(-1, 25)$ and Y.

- (a) Find the numerical value of P(|X 2| 1 < 0).
- (b) Find all possible values for an integer m such that $P(X^2 \le m^2) \le 0.4$
- (c) Let f(v) be the PDF of Y, given by

$$f(v) = \frac{K}{\sqrt{2\pi}} \exp(-\frac{v^2}{2}), \quad v \ge 0$$

and 0 for v < 0. Find the value of the constant K.

3. [Scaling Rule for pdfs]

Consider the random variable X which has the pdf given by

$$f_X(u) = \begin{cases} 1 & -0.5 \le u \le 0, \\ 1 - u & 0 \le u \le 1. \end{cases}$$

Then, we define random variables Y and Z such that Y = 2X - 3 and Z = -X + 2.

- (a) Find E[Y] and Var(Y).
- (b) Find E[Z] and Var(Z).
- (c) Find and plot the pdfs of Y and Z.

4. [The Table of Q-Function]

Consider the random variable X which is assumed to have a Gaussian distribution. For each case, find the expected value of X, i.e., E[X], and the standard deviation σ_X .

- (a) $P(X \le 6.20) = 0.9032$, and $P(X \ge 2) = 0.40129$.
- (b) $P(X \ge 4) = P(X \le 0)$, and $P(X \ge 1) = 0.69146$.
- (c) E[(X+2)(X+1)] = 7 and E[X(X+1)] = 5.

5. [Communication in Gaussian Noise]

A wireless communication system consists of a transmitter and a receiver. The transmitter sends a signal x, and the receiver observes

$$Y = x + Z,$$

where Z is a noise term, modeled as a Gaussian random variable with mean $\mu_Z = 0$ and variance $\sigma_Z^2 = 4$.

- (a) Suppose the transmitted signal is x = 2. What is the pdf of the received signal Y?
- (b) Now suppose the transmitted signal can be either x = -2 or x = 2. The receiver uses the following decoding rule: if Y > 0, it declares that x = 2; if $Y \le 0$, it declares that x = -2. Assuming that the transmitter sends -2 or +2 with probability 1/2 each, what is the receiver's error probability?
- (c) Now suppose the transmitted signal x can be chosen from three possible values: x = -4, x = 0 and x = 4. The receiver now uses the following decoding rule: if Y < -2, it declares x = -4, if Y > 2, it declares x = 4, and otherwise it declares x = 0. Assuming the transmitter sends each possible symbol with probability 1/3, what is the receiver's error probability?