University of Illinois **Spring 2024**

ECE 313: Problem Set 8: Solutions

Due: Friday, March 22 at 7 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.3 - 3.5

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. **Please write down your work and derivations. An answer without justification as of how it is found will not be accepted.** You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. [**Customer support center**]

   (a) Let X denote the number of calls during the interval of 2 minutes. As described in the course notes, the event of interest follows a Poisson distribution with mean $\lambda \times 2 = 5 \times 2 = 10$, and thus we get:

   $$\Pr(X = 4) = \frac{10^4}{4!} e^{-10} \approx 0.0189.$$

   (b) Let Y denote the number of calls during the interval of one minute. Then

   $$\begin{aligned}\Pr(Y \geq 3) =& 1 - \Pr(X < 3) \\ =& 1 - (\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)) \\ =& 1 - \left(e^{-5} + 5e^{-5} + \frac{5^2}{2!} e^{-5}\right) \\ =& 1 - \frac{37e^{-5}}{2} \approx 0.875.\end{aligned}$$

2. [**Poisson Process Intervals**]

   (a) Each "step" in a Poisson process has rate λ. We are counting successes over 5 steps, so we have a Poisson distribution with rate 5λ. Therefore

   $$P(N_5 = 7) = e^{-5\lambda} \frac{(5\lambda)^7}{7!}$$

(b) $N_8 - N_3$ is also counting successes over 5 steps. Therefore

$$P(N_8 - N_3 = 7) = e^{-5\lambda}\frac{(5\lambda)^7}{7!}$$

The expected value of a Poisson random variable over five units of time is

$$E[N_8 - N_3] = 5\lambda$$

(c) If we can break this problem into non-overlapping intervals, we can take advantage of their statistical independence. To this end, we can think of the properties of $N_8 - N_5$, $N_5 - N_4$, and $N_4 - N_3$. We know that there were 5 successes in $N_5 - N_4$. For there to have been 7 total successes gives us 3 possible cases:

i. $N_8 - N_5 = 0$, $N_4 - N_3 = 2$
ii. $N_8 - N_5 = 1$, $N_4 - N_3 = 1$
iii. $N_8 - N_5 = 2$, $N_4 - N_3 = 0$

Therefore the probability of the original event can be expressed as the union of these three (disjoint) events.

$$\begin{aligned}
P(N_8 - N_3 = 7 | N_5 - N_4 = 5) &= P(N_8 - N_5 = 0 \cap N_4 - N_3 = 2) \\
&+ P(N_8 - N_5 = 1 \cap N_4 - N_3 = 1) \\
&+ P(N_8 - N_5 = 2 \cap N_4 - N_3 = 0) \qquad (1) \\
\\
&= P(N_8 - N_5 = 0)P(N_4 - N_3 = 2) \\
&+ P(N_8 - N_5 = 1)P(N_4 - N_3 = 1) \\
&+ P(N_8 - N_5 = 2)P(N_4 - N_3 = 0) \qquad (2) \\
\\
&= (e^{-3\lambda}\frac{(3\lambda)^0}{0!})(e^{-\lambda}\frac{(\lambda)^2}{2!}) \\
&+ (e^{-3\lambda}\frac{(3\lambda)^1}{1!})(e^{-\lambda}\frac{(\lambda)^1}{1!}) \\
&+ (e^{-3\lambda}\frac{(3\lambda)^2}{2!})(e^{-\lambda}\frac{(\lambda)^0}{0!}) \qquad (3) \\
\\
&= e^{-4\lambda}(\frac{\lambda^2}{2} + 3\lambda^2 + \frac{9\lambda^2}{2}) \qquad (4) \\
&= e^{-4\lambda}(8\lambda^2) \qquad (5)
\end{aligned}$$

You can note that this is equal to $e^{-4\lambda}(\frac{(4\lambda)^2}{2!})$.

(d) We can use Bayes' Theorem based on our previous answers.

$$
\begin{aligned}
P(N_5 - N_4 = 5 | N_8 - N_3 = 7) &= \frac{P(N_8 - N_3 = 7 | N_5 - N_4 = 5) P(N_5 - N_4 = 5)}{P(N_8 - N_3 = 7)} && (6) \\
&= \frac{e^{-4\lambda}(\frac{(4\lambda)^2}{2!}) e^{-\lambda}(\frac{(\lambda)^5}{5!})}{e^{-5\lambda}(\frac{(5\lambda)^7}{7!})} && (7) \\
&= \frac{7!}{2!5!} \frac{(4\lambda)^2 \lambda^5}{(5\lambda)^7} && (8) \\
&= \binom{7}{2} \left(\frac{4}{5}\right)^2 \left(\frac{1}{5}\right)^5 && (9) \\
&\approx 0.0043 && (10)
\end{aligned}
$$

3. **[Uniform Distributions]**

The discriminant is given by

$$\Delta = (2U)^2 - 4V = 4(U^2 - V)$$

When the equation has two real solutions, it is equivalent to say $\Delta > 0$, i.e. $U^2 > V$.

As U, V are both uniformly and independently distributed on $[0, 1]$, the probability of the event $U^2 > V$ equals the area bounded by $y = x^2$, $x = 1$, $y = 0$ in a common 2D coordinate system. That is,

$$
\begin{aligned}
P(U^2 > V) &= \int_0^1 \int_0^{x^2} dy dx && (11) \\
&= \int_0^1 x^2 dx && (12) \\
&= \frac{1}{3} && (13) \\
& && (14)
\end{aligned}
$$

4. **[Exponential Distributions I]**

(a) If $g \geq 0$,

$$
\begin{aligned}
P(G \geq g) &= 1 - P(G \leq g) && (15) \\
&= 1 - \int_0^g \lambda e^{-\lambda u} du && (16) \\
&= 1 - \lambda(-\lambda)^{-1}(e^{-\lambda u})|_0^g && (17) \\
&= 1 - (1 - e^{-\lambda g}) && (18) \\
&= e^{-\lambda g} && (19) \\
&= (3e)^{-g} && (20)
\end{aligned}
$$

If $g < 0$, $P(G \geq g) = 1$. Therefore,

$$
P(G \geq g) = \begin{cases} (3e)^{-g}, & g \geq 0 \\ 1, & g < 0. \end{cases} \quad (21)
$$

Alternatively, you may directly compute the probability as $P(G \geq g) = 1 - F_G(g)$ with $F_G(g)$ being the CDF of G, i.e.

$$F_G(g) = \begin{cases} 1 - e^{-\lambda g}, & g \geq 0 \\ 0, & g < 0. \end{cases}$$

to get the same result.

(b)

$$\begin{aligned} P(G > 4 | G > 2) &= \frac{P(G > 4 \cap G > 2)}{P(G > 2)} \\ &= \frac{P(G > 4)}{P(G > 2)} \\ &= \frac{(3e)^{-4}}{(3e)^{-2}} \\ &= (3e)^{-2} \\ &\approx 0.01504 \end{aligned}$$

$$\begin{aligned} P(G < 4 | G > 2) &= \frac{P(G < 4 \cap G > 2)}{P(G > 2)} \\ &= \frac{P(G > 2) - P(G \geq 4)}{P(G > 2)} \\ &= \frac{(3e)^{-2} - (3e)^{-4}}{(3e)^{-2}} \\ &= 1 - (3e)^{-2} \\ &\approx 0.98496 \end{aligned}$$

$$\begin{aligned} P(G > 2 | G < 4) &= \frac{P(G < 4 \cap G > 2)}{P(G < 4)} \\ &= \frac{P(G > 2) - P(G \geq 4)}{1 - P(G \geq 4)} \\ &= \frac{(3e)^{-2} - (3e)^{-4}}{1 - (3e)^{-4}} \\ &\approx 0.01481 \end{aligned}$$

Alternative methods:

- $P(G > 4|G > 2)$ can be obtained directly by the memoryless property of exponential distribution;
- $P(G < 4|G > 2) = 1 - P(G \geq 4|G > 2) = 1 - P(G > 4|G > 2)$;
- $P(G > 2|G < 4) = \frac{P(G<4|G>2)P(G>2)}{P(G<4)}$ (Bayes Formula).

5. **[Understanding the Exponential]**

(a) According to the course notes (p.105), we know

$$E[X^4] = \frac{4!}{\lambda^4} = \frac{24}{\lambda^4}$$
$$E[X^2] = \frac{2!}{\lambda^2} = \frac{2}{\lambda^2}$$

Therefore,

$$\begin{aligned} E[X^4] - E[X^2] = 0 &\iff \frac{24}{\lambda^4} = \frac{2}{\lambda^2} \\ &\iff \lambda^2 = 12 \\ &\iff \lambda = 2\sqrt{3} \end{aligned}$$

(b)

$$\begin{aligned} P(\lceil X^3 \rceil = 8) &= P(\sqrt[3]{7} < X \leq 2) \\ &= (1 - e^{-2\lambda}) - (1 - e^{-\sqrt[3]{7}\lambda}) \\ &= e^{-\sqrt[3]{7}\lambda} - e^{-2\lambda} \end{aligned}$$

$$\begin{aligned} P(e < e^X < e^2) &= P(1 < X < 2) \\ &= (1 - e^{-2\lambda}) - (1 - e^{-\lambda}) \\ &= e^{-\lambda} - e^{-2\lambda} \end{aligned}$$

Therefore,

$$P(\lceil X^3 \rceil = 8) + P(e < e^X < e^2) = e^{-\lambda} + e^{-\sqrt[3]{7}\lambda} - 2e^{-2\lambda}$$

6. **[(Extra Credit) Exponential Distributions II]**

(a) Let X_1, X_2 be exponential random variables such that $X = \min(X_1, X_2)$. Note that for real numbers a, b, if $\min(a, b) > c$, then $a > c$ and $b > c$. Then for $c \geq 0$,

$$\begin{aligned} P(X \leq c) &= 1 - P(X > c) && (22) \\ &= 1 - P(X_1 > c \cap X_2 > c) && (23) \\ &= 1 - P(X_1 > c)P(X_2 > c) && (24) \\ &= 1 - e^{-2\lambda c} && (25) \end{aligned}$$

Again, the CDF is 0 for $c < 0$.
Note that this is just the CDF of $Exp(2\lambda)$ distribution.

(b) Let B_1, B_2, and B_3 denote the lifetime of battery 1, 2, and 3, respectively. Further, let T_1 denote the time until the first failure occurs. Then we know $T_1 = \min(B_1, B_2)$ and by (a), $T_1 \sim Exp(2\lambda)$, so

$$E[T_1] = \frac{1}{2\lambda}$$

Next, let T_2 denote the time between the first failure and the second. Without loss of generality, assume that B_1, failing at $t = t_m$, is the first battery that failed. Let ΔB

denote the time between t_m and the failure of B_2. By memoryless property, $\Delta B \sim Exp(\lambda)$ given that $B_2 > t_m$ since $P(\Delta B > t_n|B_2 > t_m) = P(B_2 > t_m + t_n|B_2 > t_m) = P(B_2 > t_n)$ (**Note: The last paragraph of course notes p.105 may help you to understand this part**). Now we can write $T_2 = \min(B_3, \Delta B)$, which also satisfies $T_2 \sim Exp(2\lambda)$ and thus gives

$$E[T_2] = \frac{1}{2\lambda}$$

Finally, we have $E[\text{device working time}] = E[T_1] + E[T_2] = \frac{1}{2\lambda} + \frac{1}{2\lambda} = \frac{1}{\lambda}$.

(c) We can use the same notation B_1, B_2, and B_3 as in (b). Here, battery 1 must have a longer lifetime than battery 2, so this time we assume B_2 failed at $t = t_m$ and is the first battery that failed. Similarly, let ΔB denote the time between t_m and the failure of B_1. For battery 1 to be the last battery that still works, $B_1 > B_2$ (i.e. battery 1 must outlast battery 2) and $\Delta B > B_3$ (i.e. after battery 2 dies, battery 1 must last longer than battery 3).

With the law of total probability, we have

$$\begin{aligned}
P(B_1 > B_2) &= \int_0^\infty P(B_1 > B_2|B_2 = k) f_{B_2}(k)dk \\
&= \int_0^\infty P(B_1 > k)\lambda e^{-\lambda k} dk \\
&= \int_0^\infty e^{-\lambda k}\lambda e^{-\lambda k} dk \\
&= \left[-\frac{1}{2}e^{-2\lambda k}\right]_0^\infty \\
&= \frac{1}{2}
\end{aligned}$$

and similarly, $P(\Delta B > B_3) = \frac{1}{2}$. These probabilities being $\frac{1}{2}$ should be intuitive through the memoryless property of exponential random variables. As $B_1 > B_2$ and $\Delta B > B_3$ are independent events,

$$P(\text{battery 1 is the last battery that still works}) = P(B_1 > B_2)P(\Delta B > B_3) = \frac{1}{4}$$