ECE 313: Problem Set 7: Solutions

Due: Friday, March 8 at 7 p.m.

Reading: ECE 313 Course Notes, Sections 2.12 - 3.2

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. Please write down your work and derivations. An answer without justification as of how it is found will not be accepted. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:

NAME NETID SECTION

PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. 5 points will be deducted for incorrectly assigned page numbers.

1. [Union bound]

(a) Let L_i denote the event that the professor is late for the *i*-th lab, and A_j denote the event that the professor is late at least once over *j* labs. Then,

$$A_{30} = L_1 \cup L_2 \cup ... \cup L_{30}$$
$$A_{200} = L_1 \cup L_2 \cup ... \cup L_{200}$$

and thus

$$P(A_{30}) \le P(L_1) + \dots + P(L_{30}) \le 30 \times (1 - 0.99) = 0.3$$

$$P(A_{200}) \le P(L_1) + \dots + P(L_{200}) \le 200 \times (1 - 0.99) = 2$$

You may further conclude that $P(A_{200}) \leq 1$ due to the properties of probability.

(b) Let B_i denote the event that the *i*-th lab will not start on time, and F_j denote the event that at least one lab will not start on time over a span of *j* labs. Then,

$$F_{10} = B_1 \cup B_2 \cup \dots \cup B_{10}$$

As the professors are late independently of each other, $P(B_i) \le 1 - 0.99 \times 0.99 = 0.0199$ and thus

$$P(F_{10}) \le P(B_1) + \dots + P(B_{10}) \le 10 \times 0.0199 = 0.199$$

2. [CDFs]

(a) Let Y be the number you choose, then X = 2Y. We then have

$$F_X(c) = P(X \le c) = P(Y \le \frac{c}{2}) = \begin{cases} 0, & c < 2a \\ \frac{c-2a}{2b-2a}, & 2a \le c \le 2b \\ 1, & c > 2b. \end{cases}$$

Note that this is just the CDF of Unif(2a, 2b) distribution. (b)

$$f_X(u) = \frac{dF_X(u)}{du} = \begin{cases} \frac{1}{2b-2a}, & 2a \le u \le 2b\\ 0, & \text{otherwise.} \end{cases}$$

This is just the PDF of Unif(2a, 2b) distribution.

(c)

$$E[X] = \int_{-\infty}^{\infty} u f_X(u) du$$
$$= \int_{2a}^{2b} \frac{u}{2b - 2a} du$$
$$= \frac{1}{2} \frac{(2b)^2 - (2a)^2}{2b - 2a}$$
$$= a + b$$

or you may directly use the expectation formula of uniform distribution to get

$$E[X] = \frac{2a+2b}{2} = a+b$$

By LOTUS,

$$E[X^{3}] = \int_{-\infty}^{\infty} u^{3} f_{X}(u) du$$

= $\int_{2a}^{2b} \frac{u^{3}}{2b - 2a} du$
= $\frac{1}{4} \frac{(2b)^{4} - (2a)^{4}}{2b - 2a}$
= $\frac{2(b^{4} - a^{4})}{b - a}$
= $2a^{3} + 2a^{2}b + 2ab^{2} + 2b^{3}$

3. [More on CDFs]

(a) The pdf should integrate to 1 over \mathbb{R} , i.e.

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_0^{\infty} c^2 e^{-5x} dx$$
$$= \frac{c^2}{5} = 1$$

which gives $c = \sqrt{5}$ or $c = -\sqrt{5}$. Note that with such c plugged into the pdf, $X \sim Exp(5)$.

(b)

$$F_X(x) = \int_{-\infty}^x f_X(u) du$$

=
$$\begin{cases} \int_0^x 5e^{-5u} du, & x \ge 0\\ 0, & \text{otherwise.} \end{cases}$$

=
$$\begin{cases} 1 - e^{-5x}, & x \ge 0\\ 0, & \text{otherwise.} \end{cases}$$

or you may get the same result using the CDF formula of exponential distribution.

(c) Recall that $P(1 < X < 3) = P(1 < X \le 3)$ (p.100 in course notes) since X is a continuous-type random variable. Therefore,

$$P(1 < X < 3) = F_X(3) - F_X(1) = e^{-5} - e^{-15}$$

or

$$P(1 < X < 3) = \int_{1}^{3} 5e^{-5u} du = e^{-5} - e^{-15}$$

4. [Optimal Location of a Fire Station]

(a) Let $X \sim \text{Unif}[0, L]$ be the location of the fire and $a \in [0, L]$ be the location of the fire station. We need to determine E[|X - a|].

$$E[|X-a|] = \int_0^L |u-a| f_X(u) du = \int_0^L \frac{|u-a|}{L} du = \int_0^a \frac{a-u}{L} du + \int_a^L \frac{u-a}{L} du = \frac{L}{2} + \frac{a^2}{L} - a$$

Taking the derivative of E[|X - a|] w.r.t. a and setting it to zero, we get:

$$\frac{\partial E[|X-a|]}{\partial a} = \frac{2a}{L} - 1 = 0 \implies a^* = \frac{L}{2}$$

The second derivative of E[|X - a|] is 2/L which is > 0, hence $a^* = L/2$ minimizes the average distance from the fire station to the fire.

(b) The average cost $C = E[(X - a^*)^2]$ can be obtained via LOTUS:

$$C = E[(X - a^*)^2] = \int_0^L \frac{(u - a^*)^2}{L} du$$
$$= \frac{L^2}{3} - La^* + a^{*2}$$
$$= \frac{L^2}{12}$$