ECE 313: Problem Set 6: Solutions

1. **[Bayes Formula]**

   (a)
$$P\{T\} = \frac{2}{3} \times \frac{1}{2} + \frac{1}{3}(1-p) = \frac{4}{9} \implies p = \frac{2}{3}.$$

   (b) Let A:"observing two H's" and B: "both coins are fair" be two events. We wish to obtain $P(B|A)$. It is easier to calculate $P(A|B)$ and $P(A|B^c)$ since these form a partition of the sample space. Hence:
$$P(A|B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}; \qquad P(A|B^c) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$
   Hence:
$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = \frac{1}{4} \times \frac{1}{3} + \frac{1}{6} \times \frac{2}{3} = \frac{7}{36}$$
   Therefore, from Bayes rule
$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} = \frac{\frac{1}{4} \times \frac{1}{3}}{\frac{7}{36}} = \frac{3}{7}$$

2. **[Chips Failing]**

   (a) Let D be the event of a randomly picked chip being found defective. Let A (B) denote the event that a randomly picked chip was manufactured by foundry A (B). Then:
$$P(D) = P(D|A)P(A) + P(D|A^c)P(A^c)$$
   with $P(A) = 0.3$ and $P(A^c) = P(B) = 0.7$. The number of bit failures in a chip manufactured by foundry A is $\text{Bi}(10^4, 5 \times 10^{-4})$ which can be approximated by Poi(5). Similarly, the distribution of the number of bit failures in foundry B's memory chip is Poi(1). Therefore,
$$P(D|A) = 1 - \sum_{k=0}^{4} \frac{5^k e^{-5}}{k!} = 0.5595; \qquad P(D|A^c) = 1 - \sum_{k=0}^{4} \frac{e^{-1}}{k!} = 0.0037$$
   Hence,
$$P(D) = P(D|A)P(A) + P(D|A^c)P(A^c) = 0.1704$$

   (b) Here we wish to find $P(A|D)$ using Bayes formula:
$$P(A|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{0.1678}{0.1704} = 0.9847.$$

3. **[Are you a Rational Bayesian?]**

(a) First, let's find $P(S = r)$ and $P(S = s)$. We know that

$$\begin{aligned} P(S = s) &= P(S = s|W = r)P(W = r) + P(S = s|W = s)P(W = s) \quad (1)\\ &= 0 + 1 \cdot 0.6 \quad (2)\\ &= 0.6 \quad (3)\end{aligned}$$

This should make sense: since the umbrella seller is perfectly honest (and all-knowing), the probability that he says the weather is sunny and the probability that the weather actually is sunny should be the same. Similarly,

$$\begin{aligned} P(S = r) &= P(S = r|W = r)P(W = r) + P(S = r|W = s)P(W = s) \quad (4)\\ &= 1 \cdot 0.4 + 0 \quad (5)\\ &= 0.4 \quad (6)\end{aligned}$$

In order to find $E[X|S = r]$, we need to find $P(W = r|S)$. When $S = r$,

$$\begin{aligned} P(W = r|S = r) &= \frac{P(S = r|W = r)P(W = r)}{P(S = r)} \quad (7)\\ &= \frac{0.4}{0.4} \quad (8)\\ &= 1 \quad (9)\end{aligned}$$

Similarly,

$$\begin{aligned} P(W = r|S = s) &= \frac{P(S = s|W = r)P(W = r)}{P(S = s)} \quad (10)\\ &= \frac{0}{0.6} \quad (11)\\ &= 0 \quad (12)\end{aligned}$$

Because X is determined entirely by the conditional probabilities we just found, we can say that $E[X|S = r]$ is 1 and $E[X|S = s]$ is 0. By the law of total expectation, we can find that

$$\begin{aligned} E[X] &= E[X|S = s]P(S = s) + E[X|S = r]P(S = r) \quad (13)\\ &= 0.4 \quad (14)\end{aligned}$$

(b) We will follow the same approach as last time and see some interestingly different results. First, let's find $P(S = r)$ and $P(S = s)$. We know that

$$\begin{aligned} P(S = s) &= P(S = s|W = r)P(W = r) + P(S = s|W = s)P(W = s) \quad (15)\\ &= 0 \quad (16)\\ & \quad (17)\end{aligned}$$

Similarly,

$$\begin{aligned} P(S = r) &= P(S = r|W = r)P(W = r) + P(S = r|W = s)P(W = s) \quad (18)\\ &= 1 \cdot 0.4 + 1 \cdot 0.6 \quad (19)\\ &= 1 \quad (20)\end{aligned}$$

In order to find $E[X|S=r]$, we need to find $P(W=r|S)$. (Note that because $P(S=s)=0$, the conditional probability $P(W=r|S=s)$ is undefined). When $S=r$,

$$\begin{aligned} P(W=r|S=r) &= \frac{P(S=r|W=r)P(W=r)}{P(S=r)} && (21)\\ &= \frac{0.4}{1} && (22)\\ &= 0.4 && (23)\end{aligned}$$

Following similar logic to part a, we can see that $E[X|S=r]=0$.
Because r is the only possible value of S, $E[X|S=r]$ completely captures the behavior of $E[X]$. So $E[X]=0$.

(c) First, let's find $P(S=r)$ and $P(S=s)$. We know that

$$\begin{aligned} P(S=s) &= P(S=s|W=r)P(W=r)+P(S=s|W=s)P(W=s) && (24)\\ &= 0+\frac{1}{3}\cdot 0.6 && (25)\\ &= 0.2 && (26)\end{aligned}$$

Similarly,

$$\begin{aligned} P(S=r) &= P(S=r|W=r)P(W=r)+P(S=r|W=s)P(W=s) && (27)\\ &= 1\cdot 0.4+\frac{2}{3}\cdot 0.6 && (28)\\ &= 0.8 && (29)\end{aligned}$$

In order to find $E[X|S=r]$, we need to find $P(W=r|S)$. When $S=r$,

$$\begin{aligned} P(W=r|S=r) &= \frac{P(S=r|W=r)P(W=r)}{P(S=r)} && (30)\\ &= \frac{0.4}{0.8} && (31)\\ &= 0.5 && (32)\end{aligned}$$

When $S=s$,

$$\begin{aligned} P(W=r|S=s) &= \frac{P(S=s|W=r)P(W=r)}{P(S=s)} && (33)\\ &= \frac{0}{0.2} && (34)\\ &= 0 && (35)\end{aligned}$$

Because X is deterministic, $E[X|S=r]$ is 1 and $E[X|S=s]$ is 0 like in part a. Finally, by the law of total expectation, we can find that

$$\begin{aligned} E[X] &= E[X|S=s]P(S=s)+E[X|S=r]P(S=r) && (36)\\ &= 0.8 && (37)\end{aligned}$$

If the seller lies optimally, it seems like David is (putting it loosely) "twice as likely" to buy an umbrella compared to when the seller is completely honest!

(d)

4. **[ML Hypothesis Testing]**

(a) The likelihood matrix for this problem is:

	$X=0$	$X=1$	$X=2$	$X=3$	$X=4$
H_0	$\underline{1/5}$	$1/5$	$1/5$	$1/5$	$\underline{1/5}$
H_1	$1/16$	$\underline{1/4}$	$\underline{3/8}$	$\underline{1/4}$	$1/16$

The underlines in the table indicate which hypothesis is chosen.

(b) The likelihood ratio is given by

$$\Lambda(k) = \frac{p_1(k)}{p_0(k)} = 5\binom{4}{k}\left(\frac{1}{2}\right)^4, \; k = 0,1,2,3,4.$$

The ML rule is therefore to declare H_1 whenever $\Lambda(X) > 1$, or in this case $X = 1,2,3$, and declare H_0 when $\Lambda(X) < 1$, or $X = 0,4$. Note that $\Lambda(X)$ cannot equal 1, and therefore we do not need to break ties.

(c)

$$p_{\text{false-alarm}} = P(\text{declare } H_1 | H_0 \text{ true}) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5},$$

and

$$p_{\text{miss}} = P(\text{declare } H_0 | H_1 \text{ true}) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}.$$

(d)

$$p_e = \pi_0 \cdot p_{\text{false-alarm}} + \pi_1 \cdot p_{\text{miss}} = \frac{1}{3}\frac{3}{5} + \frac{2}{3}\frac{1}{8} = \frac{17}{60}.$$

5. **[MAP Hypothesis Testing]**

(a) The joint probability matrix for this problem is:

	$X=0$	$X=1$	$X=2$	$X=3$	$X=4$
H_0	$\underline{1/15}$	$1/15$	$1/15$	$1/15$	$\underline{1/15}$
H_1	$1/24$	$\underline{1/6}$	$\underline{1/4}$	$\underline{1/6}$	$1/24$

The underlines in the table indicate which hypothesis is chosen.

(b) The likelihood ratio is the same as 4(b). The MAP rule is to declare H_1 whenever

$$\Lambda(X) > \frac{\pi_0}{\pi_1} = \frac{1}{2}$$

or in this case, $X = 1,2,3$, and declare H_0 when $X = 0,4$. Again we cannot have $\Lambda(X) = \frac{1}{2}$ and we do not need to worry about breaking ties.

(c) The decision made using MAP decision rule is the same as using ML decision rule, so

$$p_{\text{false-alarm}} = \frac{3}{5} \qquad\qquad p_{\text{miss}} = \frac{1}{8}.$$

Therefore

$$p_e = \frac{1}{3}p_{\text{false-alarm}} + \frac{2}{3}p_{\text{miss}} = \frac{17}{60}.$$

or directly

$$p_e = \frac{1}{15} + \frac{1}{15} + \frac{1}{15} + \frac{1}{24} + \frac{1}{24} = \frac{17}{60}$$

(d) Note that $p_e(\text{MAP}) = p_e(\text{ML})$, so the error of using MAP decision rule does not exceed the error of using ML decision rule. This does not contradict what we know from class that the MAP rule minimizes p_e for given priors on the hypotheses.