University of Illinois **Spring 2024**

ECE 313: Problem Set 6

Due: Tuesday, March 5 at 7:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 2.10 - 2.11

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. **NOTE THAT THIS HOMEWORK IS INSTEAD DUE ON TUESDAY. Please write down your work and derivations. An answer without justification as of how it is found will not be accepted.** You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.
Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. [**Bayes Formula**]
   Mike has 3 coins in his pocket, 2 of which are fair coins while the third is a biased coin with $P(H) = p \neq \frac{1}{2}$.

   (a) Mike picks a coin chosen at random from his pocket. The probability that the coin will land tails is $\frac{4}{9}$. What is the value of p?

   (b) Now let us assume that the unfair coin in Mike's pocket has $p = 1/3$. Mike picks two coins at random from his pocket, tosses each coin once, and observes two heads. What is the conditional probability that both coins are fair?

2. [**Chips Failing**]
   A semiconductor manufacturer has two fabrication facilities (also called foundries) A and B that manufacture identical memory chips. Both foundries send all their memory chips for testing to an in-house test facility which combines chips from both fabs into a single lot to test. Foundry A is older and it supplies 30% of the memory chips and foundry B is newer and it supplies the rest. Each memory chip has a storage capacity of 10^4 bits. Foundry A's memory chip has a bit failure probability of $p = 5 \times 10^{-4}$ and foundry B's memory chip has a bit failure probability of $p = 10^{-4}$. A chip with more than 4 defective bits is considered a defective chip and is discarded by the test facility. Hint: use the Poisson approximation of a Binomial distribution to solve this problem.

   (a) A test engineer picks a chip at random to test. What is the probability that the chip fails?

(b) If a randomly picked chip is found defective, what is the probability that it came from foundry A?

3. **[Are you a Rational Bayesian?]**
Consider a typical day in Los Angeles which can be either sunny denoted by state $W = s$ or rainy denoted by state $W = r$ where $P(W = r) = 0.4$ and $P(W = s) = 0.6$. In one of these days, David decided to go out and forgot to take his umbrella with him. Luckily, David saw an umbrella seller on the street. He wants to purchase an umbrella if the weather is rainy, otherwise, he does not want to buy it. On the other hand, the umbrella seller, of course, wants to sell as many umbrellas as possible. The umbrella seller tells you that it is going to be either a rainy or a sunny day with the messages $S = s$, or $S = r$, respectively. As a fellow ECE 313 student who just learned the Bayes' rule in class, David is rational Bayesian and is able to find $P(W|S)$, i.e., the conditional probability of the weather given the umbrella seller's message S. David will take either action $X = 1$ (indicating buying an umbrella), or action $X = 0$ (indicating not buying an umbrella) based on the following decision rule:

$$X = \begin{cases} 1, & P(W = r|S) \geq 0.5, \\ 0, & P(W = r|S) < 0.5. \end{cases}$$

In other words, David will purchase an umbrella if the conditional probability of rainy weather after observing umbrella seller's message S is higher than or equal to 0.5. Different from David, the umbrella seller knows the weather condition for each day. If the weather is going to be rainy, since the umbrella seller wants to maximize his profit, he always tells the weather is going to be rainy. In other words, we have:

$$P(S|W = r) = \begin{cases} 1, & \text{if } S = r, \\ 0, & \text{if } S = s. \end{cases}$$

Let us consider the umbrella seller's messaging policy when the weather is sunny.

(a) **Truthful Umbrella Seller:** In this case, when the weather is sunny, the umbrella seller has the following messaging policy:

$$P(S|W = s) = \begin{cases} 0, & \text{if } S = r, \\ 1, & \text{if } S = s. \end{cases}$$

With these given messaging policies, first find $P(S = r)$ and $P(S = s)$. Then, find the expected action of David, for a given seller's message, i.e., $E[X|S = r]$ and $E[X|S = s]$. Finally, find $E[X]$, which is proportional to the expected utility of the umbrella seller.

(b) **Fully Deceptive Seller:** In this case, when the weather is sunny, the umbrella seller has the following messaging policy:

$$P(S|W = s) = \begin{cases} 1, & \text{if } S = r, \\ 0, & \text{if } S = s. \end{cases}$$

Thus, in this case, irrespective of the weather condition, the umbrella seller always tells that the weather will be rainy. With these given messaging policies, first find $P(W = r|S = r)$. How is it different than $P(W = r)$? Do you think obtaining seller's message in this case provide any additional information to David? Explain why you observed such a result. Then, find $P(S = r)$ and $P(S = s)$. Next, find the expected action of David for a given Seller's message $S = r$, i.e., $E[X|S = r]$. Finally, find $E[X]$, which is proportional to the expected utility of the umbrella seller.

(c) **Optimally Deceptive Seller:** At some point, the seller notices that being fully deceptive or completely truthful is not maximizing his utility. Perhaps, by carefully adjusting his messaging policy, seller applies the following messaging policy:

$$P(S|W = s) = \begin{cases} \frac{2}{3}, & \text{if } S = r, \\ \frac{1}{3}, & \text{if } S = s. \end{cases}$$

In this case, the seller partially tells the truth when the weather is sunny, but sometimes lies in order to maximize his own utility. With this messaging policy, find $P(S = r)$ and $P(S = s)$. Then, find the expected action of David for a given Seller's message, i.e., $E[X|S = r]$ and $E[X|S = s]$. Finally, find $E[X]$, which is proportional to the expected utility of the umbrella seller.

(d) **Optional Reading and some further optional questions:** This question is related to Bayesian Persuasion which is one of the active research areas in Economics, and also in Engineering. If you are interested in, please have a look at the first 5 pages of the following seminal work in that area: "E. Kamenica and M. Gentzkow. Bayesian persuasion. American Economic Review, 101(6):2590–2615, 2011." One of the central assumption of this work and the most of the work followed by this paper is that the information receivers are assumed to be Bayesian. As a fellow ECE 313 student, since you learned Bayes' rule, you can be considered as perfectly Bayesian. On the other hand, can we make the same assumption for the general public who may or may not have any probability background? Even if you are perfectly Bayesian, if you know that the other entity who provide information to you potentially benefits from your action, would you still behave in the same way that is desired by the information provider?

4. **[ML Hypothesis Testing]**
Consider the hypothesis testing problem in which the pmf's of the observation X under hypotheses H_0 and H_1 are given, respectively, by:

$$p_0(k) = \frac{1}{5} \text{ for } k = 0, 1, 2, 3, 4.$$

and

$$p_1(k) = \binom{4}{k} \left(\frac{1}{2}\right)^4 \text{ for } k = 0, 1, 2, 3, 4.$$

(a) Find the ML decision rule using the likelihood matrix.

(b) Confirm that you obtain the same ML rule from the likelihood ratio form.

(c) Find $p_{\text{false-alarm}}$ and p_{miss} for the ML rule.

(d) Assuming priors $\pi_0 = \frac{1}{3}$ and $\pi_1 = \frac{2}{3}$, find the average probability of error p_e for the ML rule.

5. **[MAP Hypothesis Testing]**
Consider the same hypotheses as in Problem 4. Assume priors $\pi_0 = \frac{1}{3}$ and $\pi_1 = \frac{2}{3}$.

(a) Find the MAP decision rule using the joint probability matrix.

(b) Confirm that you obtain the same MAP rule using the likelihood ratio form.

(c) Find the average probability of error p_e for the MAP rule.

(d) Compare the p_e value for the MAP rule with that of the ML rule for priors $\pi_0 = \frac{1}{3}$ and $\pi_1 = \frac{2}{3}$.