University of Illinois **Spring 2024**

ECE 313: Problem Set 5: Solutions

1. **[Maximum-likelihood Estimation]**

   (a) If $\hat{n}_{ML} < 5$, then the highest value X can be is < 9.

   (b) We want to maximize $p_X(9) = \frac{1}{n}$ by choosing the best value of n. In this case its clear that $\frac{1}{n}$ is decreasing for positive integer n. Therefore when n is as small as possible, $p_X(9)$ is as big as possible. The smallest value n can take with $X = 9$ being possible is 5. Therefore

   $$\hat{n}_{ML} = 5$$

   (c) Again, $\frac{k}{n^2}$ is a decreasing function for positive integers n. The smallest value n can take with $X = 9$ being possible is 5. Therefore

   $$\hat{n}_{ML} = 5$$

   (d) Unlike the other functions, this pmf is not uniformly decreasing for positive integers when $k = 9$. When we compute the values, we see that

n	5	6	7	8	9	10	11	12	13
$P(X = 9 \mid n)$	0.04	0.0833	0.1020	0.1094	0.1111	0.11	0.1074	0.1042	0.1006

   Numerically, it seems like $n = 9$ gives the highest likelihood estimate. We can say that

   $$\hat{n}_{ML} = 9$$

   We can also verify this by modeling $p_X(9) = \frac{2n-9}{n^2}$ as the continuous function $L(x) = \frac{2x-9}{x^2}$, where x can be any real number. We can find that $L'(x) = 2(\frac{9-x}{x^3})$, which equals 0 when $x = 9$ and is negative for any $x > 9$.

2. **[Markov Inequality]**

   (a) X is non-negative, so we can directly apply the Markov Inequality. The expected value of a discrete uniform random variable is $\frac{a+b}{2}$, so in this case $E[X] = 5$. Therefore

   $$P(X \geq c) \leq \frac{5}{c}$$

   For the rest of these problems, we will implicitly note that X is non-negative.

   Now, the probability that $X > c$ can be found by enumerating the cases. There are multiple ways to express the result, but one example is

   $$P(X \geq c) = \begin{cases} 1 & \text{if } c \leq 1 \\ \frac{10-\lceil c \rceil}{9} & \text{if } 1 < c \leq 9 \\ 0 & \text{if } c > 9 \end{cases}$$

   We can see that, for positive values of c, the Markov bound always holds. Note that the bound only provides nontrivial information when $c > E[X]$.

(b) The expected value of a binomial random variable is np. Therefore $E[X] = 5$, and

$$P(X \geq c) \leq \frac{5}{c}$$

We know that $P(X = k) = \binom{10}{k}(0.5)^{10}$. A little more work can show that

$$P(X \geq c) = \begin{cases} 1 & \text{if } c \leq 0 \\ \frac{1023}{1024} & \text{if } 0 < c \leq 1 \\ \frac{1013}{1024} & \text{if } 1 < c \leq 2 \\ \frac{121}{128} & \text{if } 2 < c \leq 3 \\ \frac{53}{64} & \text{if } 3 < c \leq 4 \\ \frac{319}{512} & \text{if } 4 < c \leq 5 \\ \frac{193}{512} & \text{if } 5 < c \leq 6 \\ \frac{11}{64} & \text{if } 6 < c \leq 7 \\ \frac{7}{128} & \text{if } 7 < c \leq 8 \\ \frac{11}{1024} & \text{if } 8 < c \leq 9 \\ \frac{1}{1024} & \text{if } 9 < c \leq 10 \\ 0 & \text{if } c > 10 \end{cases}$$

and all these values are bounded by the corresponding Markov bound $\frac{5}{c}$ for positive c.

(c) The expected value of a binomial random variable is $\frac{1}{p}$. Therefore $E[X] = 5$, and

$$P(X \geq c) \leq \frac{5}{c}$$

We know that $P(X = c) = (0.2)(0.8)^{c-1}$. We can use the geometric series formula

$$\sum_{k=1}^{n} r^{k-1} = \frac{1 - r^n}{1 - r}$$

to help simplify.

In our case, we have

$$\begin{aligned} P(X \geq c) &= 1 - P(X < c) && (1) \\ &= 1 - \sum_{k=1}^{\lceil c \rceil - 1} (\frac{1}{5})(\frac{4}{5})^{k-1} && (2) \\ &= 1 - \frac{1}{5} \sum_{k=1}^{\lceil c \rceil - 1} (\frac{4}{5})^{k-1} && (3) \\ &= 1 - \frac{1}{5} \frac{1 - (\frac{4}{5})^{\lceil c \rceil - 1}}{\frac{1}{5}} && (4) \\ &= 1 - (1 - (\frac{4}{5})^{\lceil c \rceil - 1}) && (5) \\ &= (\frac{4}{5})^{\lceil c \rceil - 1} && (6) \\ & && (7) \end{aligned}$$

Therefore,

$$P(X \geq c) = \begin{cases} (\frac{4}{5})^{\lceil c \rceil - 1} & \text{if } c \geq 1 \\ 1 & \text{otherwise} \end{cases}$$

We know that the Markov inequality bounds the actual probability when

$$P(X \geq c) \leq \frac{E[X]}{c}$$

. We can use the identities provided in the Campuswire hint to help us.

$$\begin{aligned} P(X \geq c) &= (\frac{4}{5})^{\lceil c \rceil - 1} && (8) \\ &= (1 - \frac{1}{5})^{\lceil c \rceil - 1} && (9) \\ &\leq \exp((-\frac{1}{5})(\lceil c \rceil - 1)) && (10) \\ &\leq \frac{1}{1 + \frac{1}{5}(\lceil c \rceil - 1)} && (11) \\ &= \frac{5}{4 + \lceil c \rceil} && (12) \\ &\leq \frac{5}{c} && (13) \end{aligned}$$

3. **[Chebychev inequality]**

(a) With the pmf of X, we can compute that $E[X] = 0$ and $\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{16+9+4+1+0+1+4+9+16}{9} = \frac{20}{3}$. Therefore the Chebychev inequality here is

$$P(|X| \geq c) \leq \frac{20}{3c^2}$$

In the meantime, the exact probabilities are

$$P(|X| \geq c) = \begin{cases} 1 & \text{if } c \leq 0 \\ \frac{10 - 2\lceil c \rceil}{9} & \text{if } 0 < c \leq 4 \\ 0 & \text{if } c > 4 \end{cases}$$

and thus we can check that for any positive c, the exact probability is smaller than or equal to the bound given by Chebychev inequality.

(b) Here we have $E[X] = np = 5$ and $\text{Var}(X) = np(1-p) = \frac{5}{2}$. By Chebychev inequality,

$$P(|X - 5| \geq c) \leq \frac{5}{2c^2}$$

In the meantime, we can compute the exact probabilities as

$$P(|X - 5| \geq c) = \begin{cases} 1 & \text{if } c \leq 0 \\ \frac{193}{256} & \text{if } 0 < c \leq 1 \\ \frac{11}{32} & \text{if } 1 < c \leq 2 \\ \frac{7}{64} & \text{if } 2 < c \leq 3 \\ \frac{11}{512} & \text{if } 3 < c \leq 4 \\ \frac{1}{512} & \text{if } 4 < c \leq 5 \\ 0 & \text{if } c > 5 \end{cases}$$

For any positive c, the bound given by Chebyshev inequality holds.

(c) No, because the Markov inequality only applies for non-negative random variables. Alternatively, one may observe that for (a), $E[X] = 0$, so the Markov bound is 0 as well. However, $P(X \geq c) \leq 0$ does not always hold (e.g. when $c = 0$).

4. **[Confidence Interval]**

According to (2.15) in course notes,

$$P\left(p \in \left(\hat{p} - \frac{a}{2\sqrt{n}}, \hat{p} + \frac{a}{2\sqrt{n}}\right)\right) \geq 1 - \frac{1}{a^2}$$

where for this problem,

$$\begin{aligned} 1 - \frac{1}{a^2} &= 0.98 \\ \frac{a}{2\sqrt{n}} &\leq 0.01 \end{aligned}$$

Then we can conclude that $n \geq 125000$.