University of Illinois **Spring 2024**

ECE 313: Problem Set 5

Due: Friday, February 23 at 7:00:00 p.m.
Reading: *ECE 313 Course Notes,* Section 2.8, 2.9.

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. **Please write down your work and derivations. An answer without justification as of how it is found will not be accepted.** You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. [**Maximum-likelihood Estimation**]
   Let X denote a discrete random variable that takes on odd integer values 1, 3, ..., $2n-1$. The value of n is unknown, and we wish to find its maximum-likelihood estimate $\hat{n}_{ML}$ from the observation that X had value 9 on a trial of the experiment.

   (a) Explain why $\hat{n}_{ML}$ must be 5 or more.

   (b) Suppose that X has the pmf given by

   $$p_X(k) = \begin{cases} \frac{1}{n}, & k = 1, 3, \ldots, 2n-1, \\ 0, & \mathit{otherwise}. \end{cases}$$

   What is $\hat{n}_{ML}$ in this case?

   (c) Suppose that X has the pmf given by

   $$p_X(k) = \begin{cases} \frac{k}{n^2}, & k = 1, 3, 5, \ldots, 2n-1, \\ 0, & \mathit{otherwise}. \end{cases}$$

   What is $\hat{n}_{ML}$ in this case?

   (d) Suppose that X has the pmf given by

   $$p_X(k) = \begin{cases} \frac{2n-k}{n^2}, & k = 1, 3, 5, \ldots, 2n-1, \\ 0, & \mathit{otherwise}. \end{cases}$$

Compute the value of $p_X(9)$ for $n = 5, 6, 7, \ldots$ to find the maximum-likelihood estimate $\hat{n}_{ML}$ numerically.

2. **[Markov Inequality]**
Find the Markov inequality and the exact probability for the event $\{X \geq c\}$ as a function of c for the following cases and show that the exact probability is smaller than or equal to Markov inequality:

    (a) X is a random variable with the pmf given by

    $$p_X(k) = \begin{cases} \frac{1}{9}, & 1 \leq k \leq 9, \\ 0, & otherwise. \end{cases}$$

    (b) X is a Binomial random variable with parameters $n = 10$ and $p = 0.5$.

    (c) X is a geometric random variable with parameter $p = 0.2$.

3. **[Chebychev inequality]**
Find the Chebychev inequality and the exact probability for the event $\{|X - \mu| \geq c\}$ as a function of c for the following cases and show that the exact probability is smaller than or equal to Chebychev inequality:

    (a) X is a random variable with the pmf given by

    $$p_X(k) = \begin{cases} \frac{1}{9}, & -4 \leq k \leq 4, \\ 0, & otherwise. \end{cases}$$

    (b) X is a Binomial random variable with parameters $n = 10$ and $p = 0.5$.

    (c) For the random variable X given in part(a), can you use Markov inequality to find a bound on the probability for the event $\{X \geq c\}$? Justify your answer.

4. **[Confidence Interval]**
Let p be the probability of failure of an item. Suppose that n items are tested, and X fail. Consider the point estimator $\hat{p} = \frac{X}{n}$. Find n so that p is to be estimated to within 0.01 with 98% confidence.