ECE 313: Problem Set 4: Solutions

1. **[Binomial and Poisson]**

   (a) Let $q = 1-p$. We can use the binomial series to help us solve $P(X \text{ is even}) + P(X \text{ is odd})$ and $P(X \text{ is even}) - P(X \text{ is odd})$. We have

$$\begin{aligned}
&P(X \text{ is even}) + P(X \text{ is odd}) \\
&= \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \binom{n}{3} p^3 q^{n-3} + \dots \\
&= (q+p)^n = 1
\end{aligned}$$

   and

$$\begin{aligned}
&P(X \text{ is even}) - P(X \text{ is odd}) \\
&= \binom{n}{0} p^0 q^n - \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} - \binom{n}{3} p^3 q^{n-3} + \dots \\
&= \binom{n}{0} (-p)^0 q^n + \binom{n}{1} (-p)^1 q^{n-1} + \binom{n}{2} (-p)^2 q^{n-2} + \binom{n}{3} (-p)^3 q^{n-3} + \dots \\
&= (q-p)^n
\end{aligned}$$

   Therefore,

$$P(X \text{ is odd}) = \frac{(q+p)^n - (q-p)^n}{2} = \frac{1-(1-2p)^n}{2}$$

   (b) We have

$$\begin{aligned}
P(Y \text{ is even}) + P(Y \text{ is odd}) &= e^{-\lambda}\frac{\lambda^0}{0!} + e^{-\lambda}\frac{\lambda^1}{1!} + e^{-\lambda}\frac{\lambda^2}{2!} + e^{-\lambda}\frac{\lambda^3}{3!} + \dots \\
&= e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} \\
&= e^{-\lambda}e^{\lambda} = 1
\end{aligned}$$

   and

$$\begin{aligned}
P(Y \text{ is even}) - P(Y \text{ is odd}) &= e^{-\lambda}\frac{\lambda^0}{0!} - e^{-\lambda}\frac{\lambda^1}{1!} + e^{-\lambda}\frac{\lambda^2}{2!} - e^{-\lambda}\frac{\lambda^3}{3!} + \dots \\
&= e^{-\lambda}\frac{(-\lambda)^0}{0!} + e^{-\lambda}\frac{(-\lambda)^1}{1!} + e^{-\lambda}\frac{(-\lambda)^2}{2!} + e^{-\lambda}\frac{(-\lambda)^3}{3!} + \dots \\
&= e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!} \\
&= e^{-\lambda}e^{-\lambda} = e^{-2\lambda}
\end{aligned}$$

   Therefore,

$$P(Y \text{ is odd}) = \frac{1-e^{-2\lambda}}{2}$$

(c) As stated in (2.9) in the course notes,

$$(1-\frac{\lambda}{n})^n \to e^{-\lambda} \text{ as } n \to \infty$$

Therefore,

$$\begin{aligned}
P(X \text{ is odd}) &= \frac{1-(1-2p)^n}{2} \\
&= \frac{1-(1-\frac{2\lambda}{n})^n}{2} \\
&\to \frac{1-e^{-2\lambda}}{2} \\
&= P(Y \text{ is odd}) \text{ as } n \to \infty
\end{aligned}$$

2. **[Waiting for Success]**

(a) Since $X \sim Geo(p)$, using the memoryless property of geometric random variables, we have

$$E[X|X>10] = 10 + E[X] = 10 + \frac{1}{p} = 14.$$

Alternative Approach: As $(X=x)\cap(X>10) = \begin{cases} \Phi, \text{if } x \leq 10 \\ X=x, \text{if } x > 10 \end{cases}$, according to the definition of conditional expectation,

$$\begin{aligned}
E[X|X>10] &= \sum_{x=1}^{\infty} xP(X=x|X>10) \\
&= \sum_{x=1}^{10} 0 + \sum_{x=11}^{\infty} x\frac{P(X=x)}{P(X>10)} \\
&= \sum_{x=11}^{\infty} x\frac{(1-p)^{x-1}p}{(1-p)^{10}} \\
&= \sum_{x=11}^{\infty} x(1-p)^{x-11}p \\
&= \sum_{k=1}^{\infty}(10+k)(1-p)^{k-1}p \\
&= 10\sum_{k=1}^{\infty}(1-p)^{k-1}p + \sum_{k=1}^{\infty} k(1-p)^{k-1}p \\
&= 10 + \frac{1}{p} = 14
\end{aligned}$$

(b) By LOTUS,

$$\begin{aligned}
E[\sin(\frac{[2X+1]\pi}{2})] &= \sum_{x=1}^{\infty} \sin(\frac{[2x+1]\pi}{2}) P(X=x) \\
&= \sum_{x=1}^{\infty} (-1)^x (1-p)^{x-1} p \\
&= \frac{p}{1-p} \sum_{x=1}^{\infty} (p-1)^x \\
&= \frac{p}{1-p} \cdot (\frac{1}{1-(p-1)} - 1) = -\frac{1}{7}
\end{aligned}$$

3. **[Bins and Balls]**

(a) Let R_2 be the event where the first two balls you draw are red. Let R_L be the event where the last ball you draw is red. We can also note that $P(R_2 \cap R_L) = P(R_3)$, where R_3 denotes the event that the first three balls you draw are red.

In general, the probability that the first two balls you draw are red is $\frac{\binom{r}{2}}{\binom{r+b}{2}}$, which is only defined when $r \geq 2$. This also simplifies to $\frac{r(r-1)}{(r+b)(r+b-1)}$.

If $r = 2$, then obviously $P(R_L|R_2) = 0$. Otherwise,

$$\begin{aligned}
P(R_L|R_2) &= \frac{P(R_L \cap R_2)}{P(R_2)} && (1) \\
&= \frac{P(R_3)}{P(R_2)} && (2) \\
&= \frac{\frac{\binom{r}{3}}{\binom{r+b}{3}}}{\frac{r(r-1)}{(r+b)(r+b-1)}} && (3) \\
&= \frac{\frac{r(r-1)(r-2)}{(r+b)(r+b-1)(r+b-2)}}{\frac{r(r-1)}{(r+b)(r+b-1)}} && (4) \\
&= \frac{(r-2)}{(r+b-2)} && (5)
\end{aligned}$$

So,

$$P(R_L|R_2) = \begin{cases} 0 & \text{if } r \leq 2 \\ \frac{r-2}{r+b-2} & \text{if } r > 2 \end{cases}$$

(b) Let B_1 be the event that the first ball you draw is blue, and let N_k be the event that k of the n balls you drew were blue. Then we need to find $P(B_1|N_k)$. This is equivalent to $\frac{P(B_1 \cap N_k)}{P(N_k)}$.

$P(N_k) = 0$ if $b < k$. Otherwise, assuming that we are equally likely to draw any ball, it is $\frac{\binom{b}{k}\binom{r}{n-k}}{\binom{r+b}{n}}$.

Again, $P(B_1 \cap N_k) = 0$ if $b < k$. Otherwise, the probability that the first ball you drew is blue, and that you drew k blue balls, is similar to what we did before. In this case, we have a $\frac{b}{b+r}$ chance for the first ball we draw to be blue. After that, we have have $n-1$

picks left, with $b-1$ blue balls and r red balls to pick from. So in total, the probability becomes $\frac{b}{b+r} \cdot \frac{\binom{b-1}{k-1}\binom{r}{n-k}}{\binom{b+r-1}{n-1}}$.

Putting these two terms in a fraction and simplifying gives $\frac{k}{n}$.

Therefore

$$P(B_1|N_k) = \begin{cases} 0 & \text{if } k > b \\ \frac{k}{n} & \text{if } k \leq b \end{cases}$$

4. **[More on Throwing Dice]**

   (a) There are 6 outcomes where E is satisfied: (1,6),(2,5),(3,4), and their opposites. Therefore $P(E) = \frac{1}{6}$. Also, $P(F) = \frac{1}{6}$. The intersection between these two events is one outcome, (4,3). Therefore because $P(E \cap F) = P(E)P(F)$, the two events are independent.

   Like for F, there are 6 outcomes where G is satisfied. The intersection between E and G is again (4,3). Therefore because $P(E \cap G) = P(E)P(G)$, the two events are independent.

   (b) Another way to show independence is to show that $P(E|F \cap G) = P(E)$. If we know that F and G are both satisfied, we know that the outcome must have been (4,3). So $P(E|\text{the roll was (4,3)})$ is obviously 1. However, $P(E) = \frac{1}{36}$ as we previously showed. So E and $F \cap G$ are not independent.

   (c) False by definition.

   (d) Let A and B be two events. Mutually exclusivity implies that $P(A \cap B) = 0$. If A and B are independent, then $0 = P(A \cap B) = P(A)P(B)$, meaning that either $P(A)$ or $P(B)$ equals 0. There are many possible solutions. A trivial example that meets this condition is:

   Let A be the event where the sum of the two dice is 0.

   Let B be the event where the sum of the two dice is 1.

   These are both mutually exclusive and independent events.

5. **[Bumped From The Flight]**

   (a) The mean of a Binomial distribution with $n = 105$ trails and probability of success $p = 0.9$ is

   $$E[X] = np = 94.5$$

   (b)

   $$\begin{aligned} P(X \leq 100) &= 1 - P(X > 100) \\ &= 1 - \sum_{k=101}^{105} P(X = k) \\ &= 1 - \sum_{k=101}^{105} \binom{105}{k} 0.9^k (0.1)^{105-k} \\ &\approx 0.98328368 \end{aligned}$$

(c) The number of no-shows $Y = y$ is distributed as

$$\begin{aligned} P(Y = y) &= P(X = 105 - y) \\ &= \binom{105}{105 - y} 0.9^{105-y} 0.1^{105-(105-y)} \\ &= \binom{105}{y} 0.1^{y} 0.9^{105-y} \end{aligned}$$

where we used

$$\binom{105}{105 - y} = \binom{105}{y}$$

Hence, $Y \sim Binomial(105, 0.1)$.

(d) We can use the Poisson approximation for a Binomial distribution when n is large, p is small and $\lambda = np$. For the Binomially distributed random variable Y, we can approximate it by using a Poisson distribution with $\lambda = 105 * 0.1 = 10.5$,

$$\begin{aligned} P(Y \geq 5) &= 1 - P(Y < 5) \\ &= 1 - \sum_{k=0}^{4} P(Y = k) \\ &= 1 - \sum_{k=0}^{4} \frac{\lambda^k e^{-\lambda}}{k!} \\ &= 0.97890643 \end{aligned}$$

The Poisson approximation is very close to the exact value and differs by less than 0.5%