ECE 313: Problem Set 3: Solutions

1. **[Conditional Probability]**

   (a) If $Z_1 = 1$, then $S = 1$ when exclusively Z_2 or Z_3 is 1. If $Z_1 = 0$, $S = 1$ when $Z_2 = 1$ and $Z_3 = 0$.

$$\begin{aligned} P(S=1) =& P(Z_1=1, Z_2=0, Z_3=1) + P(Z_1=1, Z_2=1, Z_3=0) \\ & + P(Z_1=0, Z_2=1, Z_3=0) \\ =& P(Z_1=1)P(Z_2=0)P(Z_3=1) + P(Z_1=1)P(Z_2=1)P(Z_3=0) \\ & + P(Z_1=0)P(Z_2=1)P(Z_3=0) \\ =& \frac{2}{27} + \frac{2}{27} + \frac{4}{27} \\ =& \frac{8}{27} \end{aligned}$$

   (b)

$$\begin{aligned} P(Z_1=1|S=1) &= \frac{P(Z_1=1, S=1)}{P(S=1)} \\ &= \frac{P(Z_1=1, Z_2=0, Z_3=1) + P(Z_1=1, Z_2=1, Z_3=0)}{P(S=1)} \\ &= \frac{\frac{2}{27} + \frac{2}{27}}{\frac{8}{27}} \\ &= \frac{1}{2}. \end{aligned}$$

   (c)

$$\begin{aligned} & P(Z_2=1|S=0) \\ &= \frac{P(Z_2=1, S=0)}{P(S=0)} \\ &= \frac{P(Z_1=0, Z_2=1, Z_3=1)}{P(Z_1=0, Z_2=0, Z_3=0) + P(Z_1=0, Z_2=1, Z_3=1) + P(Z_1=1, Z_2=0, Z_3=0)} \\ &= \frac{\frac{2}{27}}{\frac{8}{27} + \frac{2}{27} + \frac{4}{27}} \\ &= \frac{1}{7}. \end{aligned}$$

2. **[Career Fair]**

   (a) Let Y be a random variable denoting the number of career fair booth visits a student must make before they receive their first invitation, including the visit that results in the invitation. Then we can say that $Y \sim \text{Geo}(p)$. Hence,

$$E[Y] = \frac{1}{p} = \begin{cases} 1.1111 & \text{A student} \\ 5 & \text{C student.} \end{cases}$$

(b) Let $Z = X_1 + X_2 + X_3 + X_4 + X_5$ where $X_i = 1$ if the i-th career booth visit results in a off-campus interview. Otherwise, $X_i = 0$. Therefore, $X_i \sim \text{Be}(p)$ with $p = 0.9$ for the A student and $p = 0.2$ for the C student, and $Z \sim \text{Bi}(5, \text{p})$. Therefore, for the probability that a student receives two interviews in five career booth visits is,

$$P(Z = 2) = \binom{5}{2}(p)^2(1-p)^3 = \begin{cases} 0.0081 & \text{A student} \\ 0.2048 & \text{C student.} \end{cases}$$

(c) We can apply properties of the geometric random variable to this situation. A student not getting an invitation in the first five trials is the same as their first invitation coming after the first five trials. As a result, we can use the sum of all probabilities of the A student getting their first invitation after the fifth booth.
P(an A student does not get an invitation in 5 trials)

$$\begin{aligned} &= \sum_{k=6}^{\infty} p(1-p)^{k-1} \\ &= (1-p)^5 \sum_{k'=0}^{\infty} p(1-p)^{k'} \\ &= (1-p)^5 = (1-0.9)^5 = 10^{-5} \end{aligned}$$

It can also be observed directly that P(an A student does not get an invitation in 5 trials) $= (1-p)^5$ by the independence of each trial, or by finding $P(Z = 0)$.
P(C gets an invitation in 5 trials) =

$$1 - (1-p)^5 = 1 - (1-0.2)^5 = 0.6723$$

(d) The A student, by the memoryless property of geometric random variables. The booths the students have already visited, and the interviews they have already received, have no influence on future booth visits.

3. **[Customer support center]**

(This problem will be revisited later in the semester, so the solutions will not be provided now.)

4. **[Probability theory and dieting]**

(a) Define the events:

$$H_i = \text{“}i\text{-th toss is heads”}$$

$$T_i = \text{“}i\text{-th toss is tails”}$$

both where $i = 1, 2, 3$. Finally, define the event:

$$E = \text{“You eat the pizza”}$$

$$\begin{aligned} P(E) &= P(H_1) + P(T_1 H_2 H_3) && (1) \\ &= \frac{2}{3} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} && (2) \\ &= \frac{22}{27} && (3) \end{aligned}$$

(b) If the first toss is heads, you definitely eat the pizza i.e. $H_1 \subset E$. Therefore $H_1 \cap E = H_1$.

$$\begin{aligned} P(H_1|E) &= \frac{P(H_1 \cap E)}{P(E)} && (4) \\ &= \frac{P(H_1)}{P(E)} && (5) \\ &= \frac{\frac{2}{3}}{\frac{22}{27}} && (6) \\ &= \frac{9}{11} && (7) \end{aligned}$$

5. **[Card problems]**

(a) Let H denote the event that we got exactly one heart. There are $52 \times 51 \times 50$ ways to draw the 3 cards (since we are drawing cards in a sequential order).

- If the heart is the first card drawn: there are $13 \times 39 \times 38$ ways.
- If the heart is the second card drawn: there are $39 \times 13 \times 38$ ways.
- If the heart is the third card drawn: there are $39 \times 38 \times 13$ ways.

Therefore,

$$P(H) = \frac{3 \times (13 \times 39 \times 38)}{52 \times 51 \times 50} = \frac{741}{1700}$$

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Alternative Approach 1: you may first (simultaneously) select 3 cards containing exactly one heart, and then multiply the number of ways to rearrange these 3 cards.

$$P(H) = \frac{\binom{13}{1}\binom{39}{2} \cdot 3!}{52 \times 51 \times 50} = \frac{741}{1700}$$

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Alternative Approach 2: let H' denote the event that we got exactly one heart but when drawing 3 cards *simultaneously.* Then $P(H) = P(H')$ since there is no restriction on which draw the heart occurs.

$$P(H) = \frac{\binom{13}{1}\binom{39}{2}}{\binom{52}{3}} = \frac{741}{1700}$$

(b) Let F denote the event that the first card drawn is a heart.

$$\begin{aligned} P(F|H) &= \frac{P(FH)}{P(H)} \\ &= \frac{\frac{13 \times 39 \times 38}{52 \times 51 \times 50}}{\frac{741}{1700}} \\ &= \frac{1}{3} \end{aligned}$$

(c) Let A denote the event "getting exactly three 2s", and B denote the event "getting at least two 2s". We have to check whether

$$P(AB) = P(A)P(B)$$

Since $A \subset B$, $AB = A$, so

$$P(AB) = P(A) = \frac{4 \times 3 \times 2}{52 \times 51 \times 50} = \frac{1}{5525}$$

and

$$P(B) = \frac{3 \times (4 \times 3 \times 48) + 4 \times 3 \times 2}{52 \times 51 \times 50} = \frac{73}{5525}$$

As $P(AB) \neq P(A)P(B)$, the two event are not independent.

Alternative Approach: let A' denote the event "getting exactly three 2s but when drawing 3 cards *simultaneously*", and B' denote the event "getting at least two 2s but when drawing 3 cards *simultaneously*". Then $P(A) = P(A')$ and $P(B) = P(B')$ for similar reasons as in (a). This also leads to

$$P(AB) = P(A) = P(A') = \frac{\binom{4}{3}}{\binom{52}{3}} = \frac{1}{5525}$$

and

$$P(B) = P(B') = \frac{\binom{4}{2}\binom{48}{1} + \binom{4}{3}}{\binom{52}{3}} = \frac{73}{5525}$$