ECE 313: Problem Set 2: Solutions

1. **[Two more poker hands]**

   (a) There are $\binom{13}{5}$ ways to select the numbers for the five cards, then 4 ways to choose the suit. Thus,

$$\begin{aligned} P(FLUSH) &= \frac{\binom{13}{5} \cdot 4}{\binom{52}{5}} \\ &= \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 4}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \\ &= \frac{33}{16660} \approx 0.00198 \end{aligned}$$

   (b) There are 13 ways to choose the number common to four of the cards. Given that choice, there are 12 ways to choose the number showing on the remaining card, and four ways to choose the suit of that card. Thus,

$$\begin{aligned} P(\text{FOUR OF A KIND}) &= \frac{13 \cdot 12 \cdot 4}{\binom{52}{5}} \\ &= \frac{13 \cdot 12 \cdot 4 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \\ &= \frac{1}{4165} \approx 0.0002401 \end{aligned}$$

2. **[Principles of Counting]**

   (a) There are $\binom{6}{3}$ ways to choose entrees and $\binom{8}{4}$ ways to choose sides, so there are

$$\binom{6}{3}\binom{8}{4} = 20 \times 70 = 1400$$

   different dinners.

   (b) In order to have 4 sides, at least one of (E_1, E_2, E_3) must be ordered. Consider different choices of these entrees:

   - If E_1, E_2, E_3 are all selected, then no more entrees are needed, and we can randomly choose 4 sides out of 8.
     \# of ways $= 1 \times \binom{8}{4} = 70$.
   - If E_1 is selected but at least one of (E_2, E_3) is not selected, then there are $\binom{5}{2} - 1 = 9$ ways to choose the set of entrees. After that, there is $\binom{4}{4} = 1$ way to choose the sides.
     \# of ways $= 9 \times 1 = 9$.
   - If E_1 is not selected and E_2, E_3 are both selected, then there are $\binom{3}{1} = 3$ ways to choose the set of entrees. After that, there is $\binom{4}{4} = 1$ way to choose the sides.
     \# of ways $= 3 \times 1 = 3$.

There is no overlap in the three cases we are considering, and we have considered every possible choice, so totally there are

$$70 + 9 + 3 = 82$$

different dinners.

3. **[A Karnaugh map for three events]**

   (a) The Karnaugh Map is shown below, where (a, b), $a, b \in \{1, 2, 3, 4\}$ means that the numbers a and b are rolled for the first and second dices, respectively.

	B	B	B^c	B^c
	C^c	C	C	C^c
A	∅	∅	(1,3) (2,4)	(1,1) (2,2) (3,1) (3,3) (4,2) (4,4)
A^c	(3,2) (4,1)	(1,4) (2,3)	(1,2) (3,4)	(2,1) (4,3)

   (b) We have

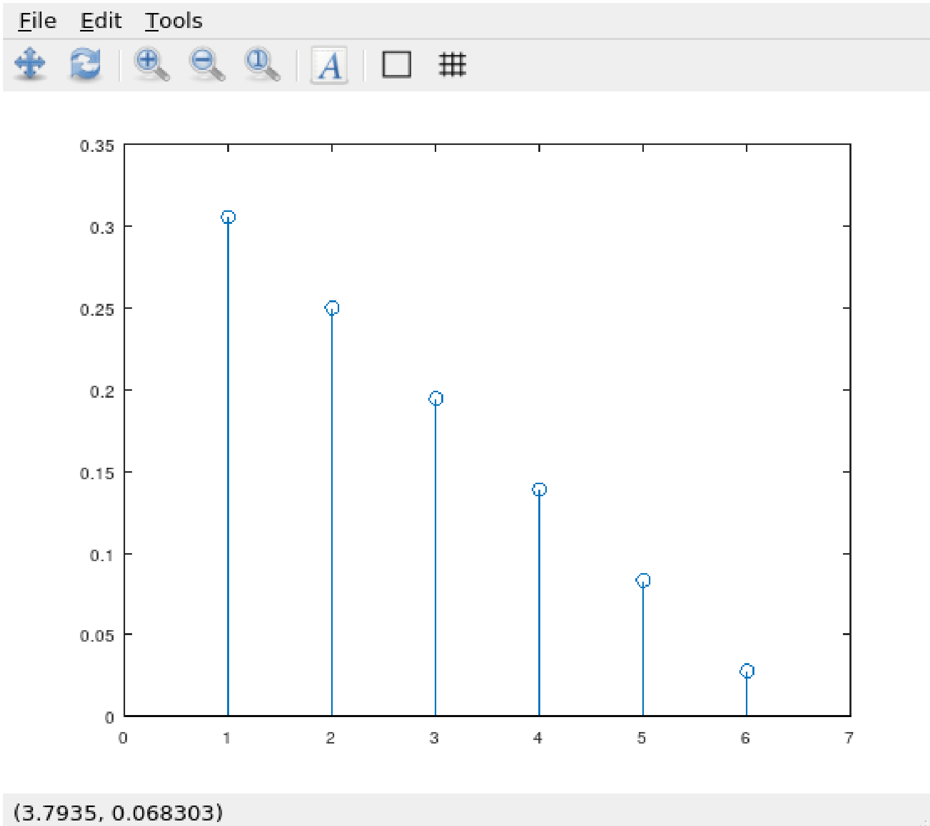
$$(A \cup B)C = (AB + A^cB + AB^c)C = ABC + A^cBC + AB^cC$$

   so

$$P((A \cup B)C) = P(ABC) + P(A^cBC) + P(AB^cC) = 0 + \frac{2}{16} + \frac{2}{16} = \frac{1}{4}$$

4. **[PMF, Mean and standard deviation]**

   (a) All 36 outcomes are equally likely. However, we can see that there are 11 ways the minimum can be 1, 9 ways the minimum can be 2 (every roll involving 2 except (1,2) and (2,1)), etc. Therefore we can say that $p_X(k) = \frac{13-2k}{36}$ for integer k such that $1 \leq k \leq 6$. Alternatively, you can write down the probabilities for each possible value of X. Below is a graph of the pmf of X:



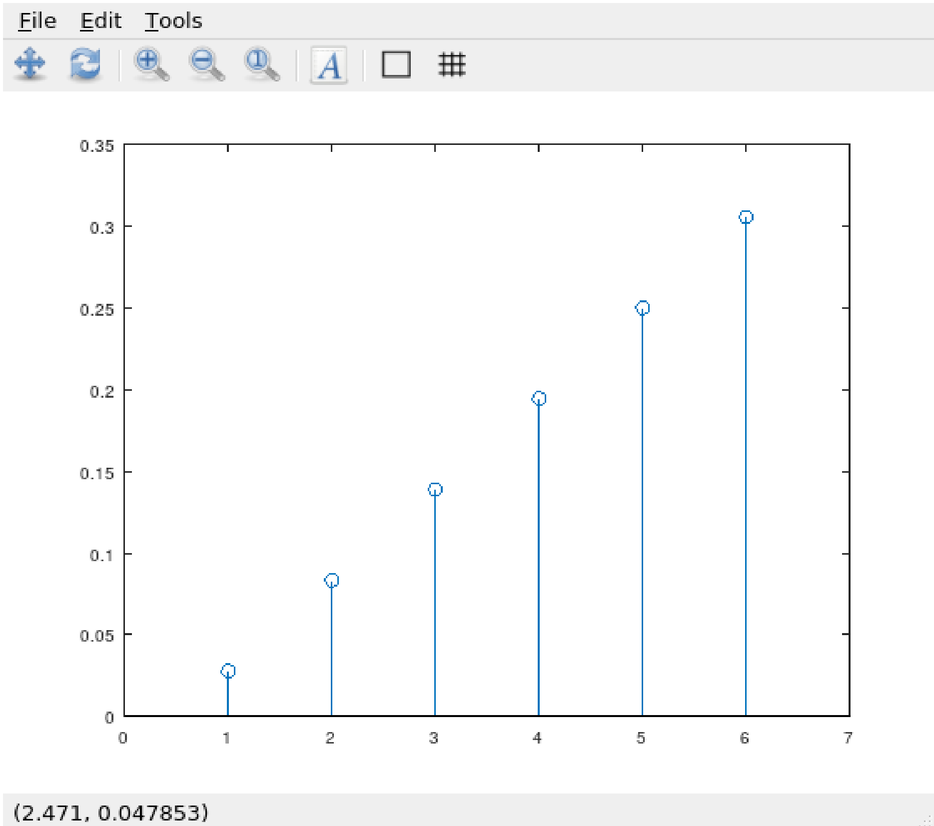
(b)

$$E[X] = 1 \cdot \frac{11}{36} + 2 \cdot \frac{9}{36} + 3 \cdot \frac{7}{36} + 4 \cdot \frac{5}{36} + 5 \cdot \frac{3}{36} + 6 \cdot \frac{1}{36} \tag{1}$$
$$= \frac{91}{36} \tag{2}$$
$$\approx 2.528 \tag{3}$$

$$E[X^2] = 1^2 \cdot \frac{11}{36} + 2^2 \cdot \frac{9}{36} + 3^2 \cdot \frac{7}{36} + 4^2 \cdot \frac{5}{36} + 5^2 \cdot \frac{3}{36} + 6^2 \cdot \frac{1}{36} \tag{4}$$
$$= \frac{301}{36} \tag{5}$$
$$\approx 8.361 \tag{6}$$

$$\sigma_X = \sqrt{\mathrm{Var}(X)} = \sqrt{E[X^2] - E[X]^2} \tag{7}$$
$$= \sqrt{\frac{301}{36} - (\frac{91}{36})^2} \tag{8}$$
$$= \sqrt{\frac{2555}{1296}} \tag{9}$$
$$\approx 1.404 \tag{10}$$

(c) All 36 outcomes are equally likely. However, we can see that there are similarities to (a) by symmetry. There are 11 ways the maximum can be 6, 9 ways the maximum can be 5 (every roll involving 5 except (5,6) and (6,5)), etc. Therefore we can say that $p_X(k) = \frac{2k-1}{36}$ for integer k such that $1 \leq k \leq 6$. Alternatively, you can write down the probabilities for each possible value of Y. Below is a graph of the pmf of Y:



(d)

$$
\begin{aligned}
E[Y] &= 1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + 3 \cdot \frac{5}{36} + 4 \cdot \frac{7}{36} + 5 \cdot \frac{9}{36} + 6 \cdot \frac{11}{36} && (11)\\
&= \frac{161}{36} && (12)\\
&\approx 4.472 && (13)
\end{aligned}
$$

$$
\begin{aligned}
E[Y^2] &= 1^2 \cdot \frac{1}{36} + 2^2 \cdot \frac{3}{36} + 3^2 \cdot \frac{5}{36} + 4^2 \cdot \frac{7}{36} + 5^2 \cdot \frac{9}{36} + 6^2 \cdot \frac{11}{36} && (14)\\
&= \frac{791}{36} && (15)\\
&\approx 21.972 && (16)
\end{aligned}
$$

$$\sigma_Y = \sqrt{\mathrm{Var}(Y)} = \sqrt{E[X^2] - E[X]^2} \tag{17}$$
$$= \sqrt{\frac{791}{36} - (\frac{161}{36})^2} \tag{18}$$
$$= \sqrt{\frac{2555}{1296}} \tag{19}$$
$$\approx 1.404 \tag{20}$$

Note that the standard deviation is the same for both X and Y.

5. **[PMF, Mean II]**

   (a) For k non-students going with you, the total cost is $10 + 50(k) + 10(4-k)$ (because you are also going). You can take between 0 and 4 non-students. $X \in \{50, 90, 130, 170, 210\}$.

   (b) Suppose you choose 0 non-student friends. In this case you all have to pay 50.
   $\# \, ways = \binom{4}{4} \cdot \binom{6}{0} = 1$
   Suppose you choose 1 non-student friend. In this case you all have to pay 90.
   $\# \, ways = \binom{4}{3} \cdot \binom{6}{1} = 24$
   Suppose you choose 2 non-student friends. In this case you all have to pay 130.
   $\# \, ways = \binom{4}{2} \cdot \binom{6}{2} = 90$
   Suppose you choose 3 non-student friends. In this case you all have to pay 170.
   $\# \, ways = \binom{4}{1} \cdot \binom{6}{3} = 80$
   Suppose you choose 4 non-student friends. In this case you all have to pay 210.
   $\# \, ways = \binom{4}{0} \cdot \binom{6}{4} = 15$
   There are 210 total ways to pick people, and all are equally likely.

$$pmf_X(k) = \begin{cases} \frac{1}{210} & k = 50 \\ \frac{24}{210} & k = 90 \\ \frac{90}{210} & k = 130 \\ \frac{80}{210} & k = 170 \\ \frac{15}{210} & k = 210 \end{cases}$$

   (c)

$$E[X] = 50 \cdot \frac{1}{210} + 90 \cdot \frac{24}{210} + 130 \cdot \frac{90}{210} + 170 \cdot \frac{80}{210} + 210 \cdot \frac{15}{210} \tag{21}$$
$$= 146 \tag{22}$$

6. **[Gambling with Dice]**

   (a) Since the die is fair,

$$p_X(2) = p_X(4) = p_X(6) = 1/6, \tag{23}$$
$$p_X(-m) = 1/2. \tag{24}$$

   (b) From the pmf we found in part (a),

$$E[X] = \frac{2}{6} + \frac{4}{6} + \frac{6}{6} - \frac{m}{2} = 2 - \frac{m}{2}.$$

Setting

$$2 - \frac{m}{2} = 0 \quad \Longrightarrow \quad m = 4.$$

(c)

$$\text{Var}(X) = E[X^2] - (E[X])^2 = 6^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + (-4)^2 \times \frac{1}{2} = \frac{52}{3},$$

since $E[X] = 0$ from part (b).