ECE 313: Problem Set 1: Solutions

1. **[Basics of Calculus]**

   (a) Let n denote a positive integer. Then, $(1-x)[1+x+x^2+\cdots+x^{n-1}] = [1+x+x^2+\cdots+x^{n-1}] - [x+x^2+x^3+\cdots x^n]$. Cancelling like terms gives

   $$(1-x)[1+x+x^2+\cdots+x^{n-1}] = 1-x^n$$

   . Because $x \neq 1$, dividing both sides by $(1-x)$ gives

   $$1+x+x^2+\cdots+x^{n-1} = \frac{1-x^n}{1-x}$$

   .

   (b) At $x=1$, the sum is $1+1+1^2+\cdots+1^{n-1}=n$. Because $\frac{1-x^n}{1-x}$ is indeterminate when $x=1$, we can apply L'Hôpital's rule:

   $$\lim_{x\longrightarrow 1}\frac{1-x^n}{1-x} = \lim_{x\longrightarrow 1}\frac{-nx^{n-1}}{-1} \tag{1}$$
   $$= n \tag{2}$$

   Therefore they are equal.

   (c) We know that $1+x+x^2+\cdots+x^{n-1} = \frac{1-x^n}{1-x}$. Then

   $$\lim_{n\longrightarrow\infty} 1+x+x^2+\cdots+x^{n-1} = \lim_{n\longrightarrow\infty}\frac{1-x^n}{1-x}$$

   . Because $|x|<1$, $\lim_{n\longrightarrow\infty} x^n = 0$. Therefore

   $$1+x+x^2+\cdots = \frac{1}{1-x}$$

   .

   (d) For the first term, we can use the product rule to find that

   $$\frac{d}{dx}\exp(-x)\sum_{k=0}^{n}\frac{x^k}{k!} = -\exp(-x)\sum_{k=0}^{n}\frac{x^k}{k!} + \exp(-x)(\frac{d}{dx}\sum_{k=0}^{n}\frac{x^k}{k!})$$

   By expanding the terms of $\sum_0^n \frac{x^k}{k!}$, we can see that

   $$(\frac{d}{dx}\sum_{k=0}^{n}\frac{x^k}{k!}) = \sum_{j=0}^{n-1}\frac{x^j}{j!}$$

   Therefore

   $$\exp(-x)(\sum_{j=0}^{n-1}\frac{x^j}{j!} - \sum_{k=0}^{n}\frac{x^k}{k!}) = -\exp(-x)\frac{x^n}{n!}$$

   The derivative of $\exp(-x)\sum_{k=0}^{\infty}\frac{x^k}{k!}$ is more straightforward. Recall that $\exp(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!}$. Therefore $\exp(-x)\sum_{k=0}^{\infty}\frac{x^k}{k!} = 1$ and the derivative is 0.

2. **[Solving definite integrals]**

   (a) Note that $x^2 - 1 < 0$ when $-1 < x < 1$. Therefore when $-1 < x < 1$, $|x^2 - 1| = 1 - x^2$. Therefore,

   $$\int_{-2}^{3} |x^2 - 1| dx = \int_{-2}^{-1} x^2 - 1 dx + \int_{-1}^{1} 1 - x^2 dx + \int_{1}^{3} x^2 - 1 dx$$

   Evaluating these three simpler integrals gives a final answer of $\frac{28}{3}$.

   (b) One way to evaluate this problem is through u-substitution. Let $u = 1 + x^3$ and $du = 3x^2 dx$. Substituting these terms into the integral gives

   $$\int_0^1 x^2(u^{10}) \frac{du}{3x^2} = \int_0^1 \frac{1}{3} u^{10} du$$

   Evaluating this simpler integral gives a final answer of $\frac{1}{33}$

   (c) One way to evaluate this problem is through integration by parts. Let $u = x$ and $dv = \exp(-x)dx$. Then by integration by parts we can argue that this integral is equivalent to

   $$(-x \exp(-x))|_0^1 - \int_0^1 - \exp(-x) dx$$

   Evaluating these terms gives a final answer of $1 - 2e^{-1}$.

   (d) This function can also be evaluated by u-substitution, but we can also note that $x^3 \sin(1 - x^4)$ is an odd function and therefore any symmetrical integral across the origin evaluates to 0.

3. **[Derivatives and integrals]**

   (a) True by chain rule.

   (b) True by chain rule.

   (c) False. The answer should be $2x \cdot g(x^2) \cdot \exp(f(x^2))$

   (d) False. The answer should be $-f(-x) + C$

   (e) False. This can be more clearly shown by differentiating both sides of the "equation."

   (f) True. We assume that $f(x) > 0$ for all x, so $\ln(f(x))$ is always defined and therefore the chain rule applies.

4. **[Defining a set of outcomes I]**

   (a) There are many different answers for this problem. Any answer must satisfy the criteria that there are three games with no ties and the loser of a previous game cannot later appear as a winner. One possible notation is $\Omega = \{(w_1, w_2, w_3) : w_1 \in \{1, 2\}, w_2 \in \{w_1, 3\}, w_3 \in \{w_2, 4\}\}$

   (b) $|\Omega| = 8$. This can be seen by either counting or recognizing that there are 3 different games with two possible winners each, so there are 2^3 possibilities.

5. **[Defining a set of outcomes II]**

(a) Let (B, R) denote the outcome that the first ball selected is blue and the second is red, and similarly for (B, B), (R, B) and (R, R). With such notation,

$$\Omega = \{(B, B),\ (B, R),\ (R, B)\}$$

Another sample space can be obtained by attaching a subscript to identically colored balls. In this case, the sample space is given by:

$$\Omega = \{(B_1, B_2), (B_1, R), (B_2, B_1), (B_2, R), (R, B_1), (R, B_2)\}$$

In this case, since both sample spaces comprise equally likely outcomes, one can confirm that the probability of obtaining a two blue balls is $1/3$ in either case. Note that the second (larger) sample space will always result in equally likely outcomes.

(b) With the same notation as in (a), we have either

$$\Omega = \{(B, B),\ (B, R),\ (R, B),\ (R, R)\}$$

or

$$\Omega = \{(X, Y) | X, Y \in \{B_1, B_2, R\}\}$$

(c) If the balls are not replaceable, the first draw will affect the second draw, as there must be a change in the balls available for the second draw. For example, if the first ball selected is red, the second must be blue.

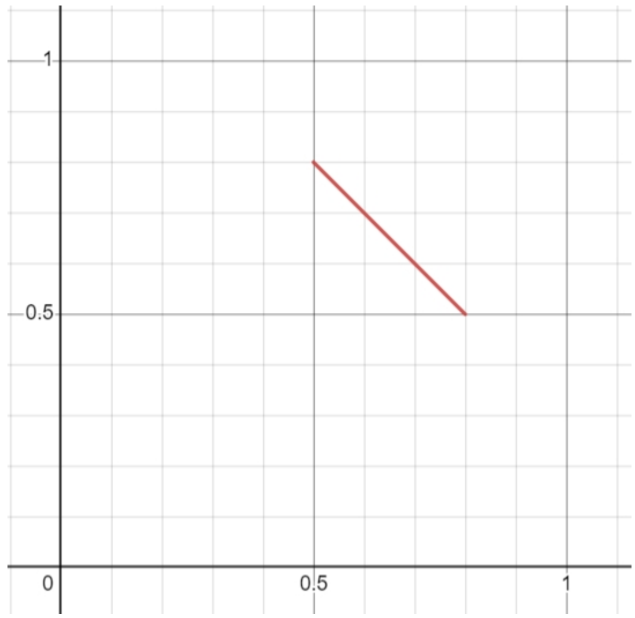
If the balls are replaceable, the first draw will not affect the second draw, as the balls available for the second draw is the same with the first draw.

6. **[Possible probability assignments I]**

By De Morgan's law, $(A \cup B)^c = A^c B^c$, so $P(A^c B^c) = 1 - P(A \cup B) = 0.2$. Besides, $P(AB) + P(AB^c) + P(A^c B) + P(A^c B^c) = 1$. Assume $P(AB^c) = m$, then we can fill the Karnaugh Map as

	B	B^c
A	0.5	m
A^c	0.3-m	0.2

Since both m and $0.3 - m$ denote probabilities, there must be $0 \leq m \leq 1$ and $0 \leq 0.3 - m \leq 1$, i.e. $0 \leq m \leq 0.3$. From the Karnaugh Map, $P(A) = 0.5 + m$, so the domain of $P(A)$ is $[0.5, 0.8]$. Additionally $P(B) = 0.8 - m$, and substituting for m gives us $P(B) = 1.3 - P(A)$. This function is plotted below, with the x-axis being $P(A)$ and the y-axis being $P(B)$:



7. **[Possible probability assignments II]**

   As a, b, c, d are elements of a sample space, the events $\{a\}, \{b\}, \{c\}, \{d\}$ are mutually exclusive. We also know that $P(\Omega) = P(\{a, b, c, d\}) = 1$. Then

$$P(\{b, c, d\}) = P(\{b\}) + P(\{c\}) + P(\{d\}) = \frac{5}{8} \tag{3}$$

$$P(\{a, b\}) = P(\{a\}) + P(\{b\}) = \frac{1}{2} \tag{4}$$

$$P(\{b, c\}) = P(\{b\}) + P(\{c\}) = \frac{1}{4} \tag{5}$$

$$P(\{a, b, c, d\}) = P(\{a\}) + P(\{b\}) + P(\{c\}) + P(\{d\}) = 1 \tag{6}$$

   Solving these equations gives

$$P(\{a\}) = \frac{3}{8}, \quad P(\{b\}) = \frac{1}{8}, \quad P(\{c\}) = \frac{1}{8}, \quad P(\{d\}) = \frac{3}{8}$$

8. **[Displaying outcomes in a two event Karnaugh map]**

   (a) The Karnaugh Map is shown below, where $(a, b),\ a, b \in \{1, 2, 3, 4\}$ means that the numbers a and b are rolled for the first and second dices, respectively.

	B	B^c
A	(1,1) (2,2)	(1,2) (2,1)
A^c	(3,3) (4,4)	(1,3) (1,4) (2,3) (2,4) (3,1) (3,2) (3,4) (4,1) (4,2) (4,3)

(b) All 16 possible outcomes are equally likely to occur, so

$$
\begin{aligned}
P(AB^c) &= \frac{2}{16} = \frac{1}{8} \\
P(A^cB) &= \frac{2}{16} = \frac{1}{8}
\end{aligned}
$$