ECE 313: Lecture 16
MAP decision rules
Union bound and its application

\[ P_{fa} = P(\text{Declare } H_1 \mid H_0) = \sum \text{ on } H_0 \text{ row but NOT under } H_1 \]

\[ P_{mis} = P(\text{Declare } H_0 \mid H_1) = \sum \]

Likelihood table

\[ p_i(k) = p(\sum x = k \mid H_i) \]

\[ \lambda(k) \geq \frac{C}{H_0} \]

\[ \lambda(k) = \frac{\sum_{i=1}^{ML} p_i(k)}{p_o(k)} \]

Prior knowledge
\[ p(H_i) = \pi_i \quad (i = 0, 1) \]

\[ p(\sum x = k_1, H_i) = p(H_i) p(\sum x = k_1 \mid H_i) \]

Joint prob
\[ p_i(k) = \pi_i p_i(k) \]

\[ \pi_0 = 0.2 \quad \pi_1 = 0.8 \]
ML: \[ \text{given } X = \mathbf{x} \]
\[ \forall i \in \{0, 1\} \quad \text{pick } H_i \text{ such that} \]
\[ \max_{i \in \{0, 1\}} \ P(\{ X = \mathbf{x} \mid H_i \}) \]

MAP (Maximum a posteriori) decision

\[ \text{given } X = \mathbf{x} \quad \text{pick } H_i \text{ that} \]
\[ \max_{i \in \{0, 1\}} \ P(\{ H_i \mid \{ X = \mathbf{x} \} \}) \]
\[ = \max_{i \in \{0, 1\}} \ P(\{ X = \mathbf{x} \mid H_i \}) \]

\[ \prod_i p_i(\mathbf{x}) \]
\[ \prod_{i=0}^{1} \frac{p_i(\mathbf{x})}{p_0(\mathbf{x})} \]

\[ \Lambda(\mathbf{x}) = \frac{p_1(\mathbf{x})}{p_0(\mathbf{x})} \]

\[ \frac{p_1(\mathbf{x})}{p_0(\mathbf{x})} \]

\[ \frac{\prod_{i=0}^{1}}{\prod_{i=0}^{1}} \cdot \frac{p_1(\mathbf{x})}{p_0(\mathbf{x})} \]
Union bound:

Context: Estimate the probability of system failure

\[ S \text{ fails if either } S_1 \text{ fails or } S_2 \text{ fails} \]

\[ P(\exists S \text{ fails}) = P(A \cup B) \]

\[ = P(A) + P(B) - P(A \cap B) \]

\[ \leq P(A) + P(B) \]

\[ \leq P(A) + P(B) \]

\[ \leq P(A) + P(B) \]

\[ \leq P(A) + P(B) \]

\[ \Rightarrow P(A) \cap P(B) \text{ small} \]

Union bound:

\[ \leq P(A \cup B) \]
\[ P(\text{whole syst fails}) = P(A \cup B) \]

if one sub syst fail:
\[ = P(A) + P(B) - P(AB) \]

\[ = p + p - p^2 \]

\[ \leq 2p \leq \text{union bound} \]

**Ex:**

\[ S_1, S_2, S_5 \]

whole syst fails if at least 5 systs fail

\[ S_3, S_4 \]

2 syst fails:

\[ P(\text{whole syst fail}) = P(F_1 F_2 U F_1 F_3 U F_1 F_4 U \ldots U F_5 F_5) \]

\[ \leq \binom{5.4}{2} p^{2} = P(F_i F_j) = P(F_i)P(F_j) \]

\[ \leq \binom{5}{2} p^{2} = 10 p^{2} \]