University of Illinois Spring 2020

ECE 313: Midterm Exam II

Wednesday, April 08, 2020

The exam consists of **6** problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. SHOW YOUR WORK. Answers without appropriate justification will receive very little credit. Reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

The solutions must be handwritten on pieces of paper or electronic notebook. Please start a new page for each sub-problem. Please scan your solutions and upload them onto Gradescope before the deadline.

1. [**15 points**]

   (a) [**5 points**] Suppose that X_1 and X_2 are random variables both with exponential distribution, X_1 having parameter λ_1 and X_2 having parameter λ_2. What must be the relationship between λ_1 and λ_2 such that $P\{X_1 \leq t\} = P\{X_2 \leq 2t\}$ for all t?
   Solution: We have

   $$P\{X_1 \leq t\} = \int_0^t \lambda_1 e^{-\lambda_1 s} ds = 1 - e^{-\lambda_1 t}$$
   $$P\{X_2 \leq 2t\} = \int_0^{2t} \lambda_2 e^{-\lambda_2 s} ds = 1 - e^{-\lambda_2 2t}. \tag{1}$$

   For these to be equal, we need $\lambda_1 = 2\lambda_2$.

   (b) [**5 points**] Let Y_1 have an exponential distribution with parameter $\lambda = \ln 2$ while Y_2 has a uniform distribution with support on the interval $[0, b]$. Find the value b such that $P\{X_1 \leq 3\} = P\{X_2 \leq 3\}$.
   Solution: We have

   $$P\{Y_1 \leq 3\} = 1 - e^{3\ln 2} = 1 - \frac{1}{8} = \frac{7}{8}$$
   $$P\{Y_2 \leq 3\} = \frac{3}{b}. \tag{2}$$

   Hence $b = \frac{24}{7}$.

   (c) [**5 points**] Suppose that $X_1, X_2, \cdots, X_n$ is a sequence of random variables in which X_j describes the total time to perform experiment j. Further suppose that X_j is a random variable having exponential distribution with parameter λ^j for some $\lambda > 1$. The experiments are such that one experiment will not begin until the previous one terminates. What is the average total time it will take to perform all n experiments in the limit $n \to \infty$? Express your answer as a function of λ.
   Solution: Since the experiments are performed sequentially, we want to compute

   $$\lim_{n\to\infty} E\left[\sum_{j=1}^{n} X_j\right] = \sum_{j=1}^{\infty} E[X_j] = \sum_{j=1}^{\infty} \frac{1}{\lambda^j} = \frac{1}{\lambda - 1}.$$

2. [**20 points**] Suppose that the number of pitches thrown in a baseball game are described by a Poisson process of rate r pitches per minute. Let N_t denote the number of pitches thrown by time t and U_j the time delay between pitch $j-1$ and j.

(a) [**6 points**] A manager plans for a pitcher to enter the game and exit after he throws 50 pitches. Let T be the random variable for the total time the pitcher is in the game. Express $E[T]$ as a function of r.
Solution: Since the time delay variables are independent, we have

$$E[T] = E\left[\sum_{i=1}^{50} U_i\right] = 50E[U_i] = \frac{50}{r}. \tag{3}$$

(b) [**7 points**] What is the pdf for the time when the fourth pitch is thrown?
Solution: For a fixed time t,

$$P\{N_t \leq 3\} = \sum_{k=0}^{3} P\{N_t = k\} = e^{-tr} \sum_{k=0}^{3} \frac{(tr)^k}{k!}.$$

Since $P\{N_t \leq 3\}$ is the probability that the fourth pitch comes after time t, we have $1 - P\{N_t \leq 3\}$ is the probability that the fourth pitch comes at time t or sooner; hence it is the cdf of the arrival time of the fourth pitch. Thus, the pdf is

$$-\frac{d}{dt} P\{N_t \leq 3\} = re^{-tr} \sum_{k=0}^{3} \frac{(tr)^k}{k!} - e^{-tr} \sum_{k=0}^{3} k \frac{t^{k-1} r^k}{k!} = e^{-tr} \frac{t^3 r^4}{3!}. \tag{4}$$

(c) [**7 points**] Each pitch in a baseball game is either a "strike" or a "ball". Let us model each pitch as a Bernoulli trial in which a "strike" is thrown with probabily s and a "ball" is thrown with probability $1 - s$, for some $0 \leq s \leq 1$. If it is known that no more than five pitches are thrown in the first four minutes of the game, what is the probability that exactly three of those pitches are "strikes"? Express your answer as a function or r and s.
Solution: For $k \leq 5$, we have

$$P[N_4 = k | N_4 \leq 5] = \frac{P[N_4 = k]}{P[N_4 \leq 5]} = \frac{\frac{(4r)^k}{k!} e^{-4r}}{\sum_{j=0}^{5} \frac{(4r)^j}{j!} e^{-4r}}.$$

We are interested in the events of throwing 3, 4, and 5 pitches and 3 of them being strikes. This is given by

$$\frac{\sum_{k=3}^{5} \frac{(4r)^k}{k!} e^{-4r} \binom{k}{3} s^3 (1-s)^{k-3}}{\sum_{j=0}^{5} \frac{(4r)^j}{j!} e^{-4r}}. \tag{5}$$

3. [**15 points**] Alice plays the same game at a casino 100 times. In the ith game, her earnings are X_i, where $P\{X_i = 1\} = 0.3$ and $P\{X_i = -1\} = 0.7$. The random variables $X_1, X_2, ..., X_{100}$ are all independent. Let $X = \sum_{i=1}^{n} X_i$ be her total earnings (which may be negative, if she loses money).

(a) [**5 points**] Let $Z_i = (X_i + 1)/2$, for $i = 1, ..., 100$ and $Z = \sum_{i=1}^{100} Z_i$. What is the distribution of Z?
Solution: Since $P\{Z_i = 1\} = P\{X_i = 1\} = 0.3$ and $P\{Z_i = 0\} = P\{X_i = -1\} = 0.7$, Z_i is Bernoulli(0.3). Hence $Z = \sum_{i=1}^{100} Z_i$ has a Binomial$(100, 0.3)$ distribution.

(b) [**5 points**] Express the event $\{X = 0\}$ in terms of Z and compute $P\{X = 0\}$.
Solution: In order for X to be 0, Alice must win exactly 50 out of the 100 games. Since Z is the number of games won, $\{X = 0\} = \{Z = 50\}$, and

$$P\{X = 0\} = P\{Z = 50\} = \binom{100}{50}(0.3)^{50}(0.7)^{50}.$$

(c) [**5 points**] Use the Gaussian approximation with continuity correction to compute $P\{|X| \geq 10\}$. Express your answer in terms of the Q function. (Hint: Express the event $\{|X| \geq 10\}$ in terms of Z.)
Solution: Since $X_i = 2Z_i - 1$, we can write

$$X = \sum_{i=1}^{100}(2Z_i - 1) = 2Z - 100.$$

Since Z is Binomial$(100, 0.3)$, $E[Z] = 100 \cdot 0.3 = 30$ and $\text{Var}(Z) = 100(0.3)(0.7) = 21$. Hence, we have

$$\begin{aligned} P\{|X| \geq 10\} &= P\{X \geq 10\} + P\{X \leq -10\} = P\{Z \geq 55\} + P\{Z \leq 45\} \\ &= P\{Z \geq 54.5\} + P\{Z \leq 45.5\} = P\left\{\frac{Z-30}{\sqrt{21}} \geq \frac{24.5}{\sqrt{21}}\right\} + P\left\{\frac{Z-30}{\sqrt{21}} \leq \frac{15.5}{\sqrt{21}}\right\} \\ &\approx Q\left(\frac{24.5}{\sqrt{21}}\right) + \Phi\left(\frac{15.5}{\sqrt{21}}\right) = Q\left(\frac{24.5}{\sqrt{21}}\right) + Q\left(\frac{-15.5}{\sqrt{21}}\right) \end{aligned}$$

4. [**15 points**] Suppose the random variable X has an exponential distribution with parameter λ. Let $Y = \min(X, X^2)$.

(a) [**7 points**] Compute the CDF of Y.
Solution: Since $X \geq 0$, $Y = \min(X, X^2) \geq 0$. Hence, for $c < 0$, $F_Y(c) = 0$. Notice that $\min(x, x^2) = x$ if $x \geq 1$ and $\min(x, x^2) = x^2$ for $0 \leq x \leq 1$. Hence, for $0 \leq c \leq 1$, we have

$$F_Y(c) = P\{\min(X, X^2) \leq c\} = P\{X^2 \leq c\} = 1 - e^{-\lambda\sqrt{c}}.$$

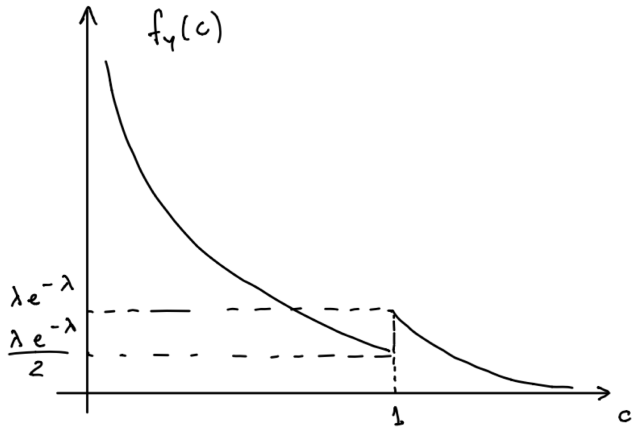
For $c > 1$, we have

$$F_Y(c) = P\{\min(X, X^2) \leq c\} = P\{X \leq c\} = 1 - e^{-\lambda c}.$$

(b) [**8 points**] Is Y a continuous-type random variable? If so, sketch its pdf, $f_Y(y)$. If not, explain why not.
Solution: Since the CDF is differentiable at every point except $c = 0$ and $c = 1$, a pdf can be obtained by differentiating the CDF (the values at $c = 0$ and $c = 1$ can be chosen arbitrarily). By differentiating the CDF, we obtain

$$f_Y(c) = \begin{cases} \frac{\lambda}{2\sqrt{c}}e^{-\lambda\sqrt{c}} & 0 < c < 1 \\ \lambda e^{-\lambda c} & c > 1 \\ 0 & \text{otherwise} \end{cases}$$

Notice that the value of $f_Y(1)$ is irrelevant and can be chosen to be $\frac{\lambda}{2}e^{-\lambda}$ or $\lambda e^{-\lambda}$.

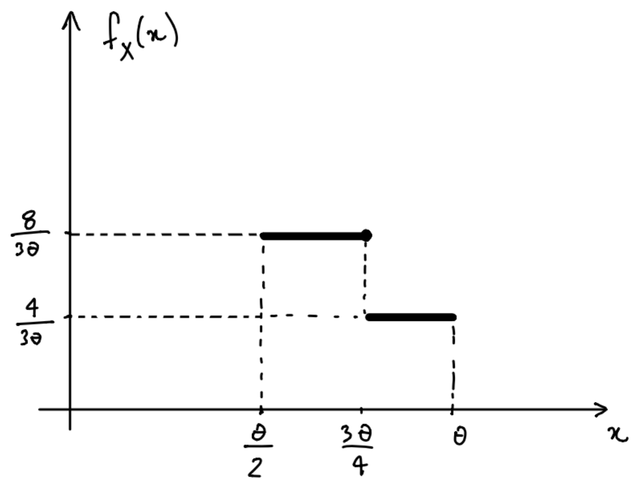


5. **[15 points]** Let X be a continuous-type random variable with pdf given by

$$f_X(x) = \begin{cases} \frac{8}{3\theta} & \theta/2 \le x \le 3\theta/4 \\ \frac{4}{3\theta} & 3\theta/4 < x \le \theta \\ 0 & \text{otherwise} \end{cases}$$

for an unknown parameter $\theta > 0$.

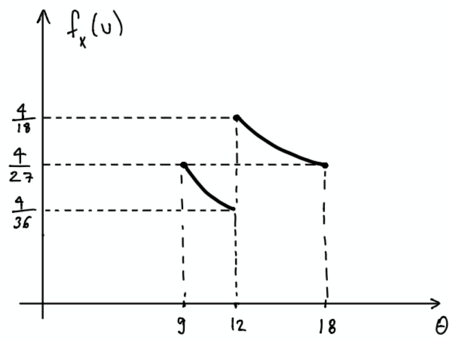
(a) [**7 points**] Sketch $f_X(x)$ for a general $\theta > 0$. Make sure to label the axes and important points, as a function of θ.
Solution:



(b) [**8 points**] You observe that $X = 9$. Compute the maximum likelihood parameter estimate $\hat{\theta}_{ML}$.
Solution: We need to find the value of θ that maximizes $f_X(u)$. To do that, we view $f_X(u)$ as a function of θ:

$$f_X(u, \theta) = \begin{cases} \frac{8}{3\theta} & 4u/3 \le \theta \le 2u \\ \frac{4}{3\theta} & u \le \theta < 4u/3 \\ 0 & \text{otherwise} \end{cases}$$

which is shown below, for $u = 9$:



Clearly, $f_X(9, \theta)$ is maximized for $\theta = 4 \cdot 9/3 = 12$. We conclude that $\hat{\theta}_{ML} = 12$.

6. **[20 points]** Company A manufactures light emitting diodes (LEDs). For each diode, the failure rate is $h(t) = \log(t+1)$ $(t \geq 0)$. Company B uses 5 LEDs from company A to build a device. If any of the diode breaks, the device will break.

   (a) [**8 points**] What is the pdf of the lifetime of a single LED?
   Solution: The CDF of the lifetime T for LED is

$$F_T(t) = 1 - e^{-\int_0^t h(s)\,ds} = 1 - e^{-\int_0^t \log(s+1)\,ds} = 1 - e^{-((t+1)\log(t+1)-t)}, \tag{6}$$

   for $t \geq 0$. Therefore, the pdf is

$$f_T(t) = \begin{cases} \log(t+1)e^{-((t+1)\log(t+1)-t)} = \log(t+1)\frac{e^t}{(t+1)^{t+1}}, & \text{if } t \geq 0 \\ 0, & \text{else} \end{cases} \tag{7}$$

   (b) [**6 points**] What is the failure rate of the device built by company B?
   Solution: Suppose $T_1, T_2, \ldots, T_5$ are the lifetime of 5 diodes. They are independent and follow the same distribution. Since a single diode failure will result in the failure of the whole device, therefore the device lifetime $T = \min(T_1, T_2, T_3, T_4, T_5)$, and

$$P(T > t) = P(T_1 > t, T_2 > t, T_3 > t, T_4 > t, T_5 > t) = \prod_{i=1}^{5} P(T_i > t) = \prod_{i=1}^{5} e^{-\int_0^T h_i(s)ds}.$$

   Therefore, $h(t) = \sum_{i=1}^5 h_i(t) = 5\log(t+1)$.

   (c) [**6 points**] This question is separate from the previous questions. Suppose the failure rate of an LED is $h(t) = at$. What is the expected lifetime of the LED?
   Solution: If $h(t) = at$ for $t \geq 0$, the CDF of the lifetime is $F_T(t) = 1 - e^{-\int_0^t asds} = 1 - e^{-\frac{at^2}{2}}$. One approach to find $E[T]$ is to first find the pdf by differentiating F_T, which yields $f_T(t) = ate^{-\frac{at^2}{2}}$. Then we have

$$E[T] = \int_0^\infty t(at)e^{-\frac{at^2}{2}}\,dt = -\,te^{-\frac{at^2}{2}}\Big|_0^\infty + \int_0^\infty e^{-\frac{at^2}{2}}\,dt \tag{8}$$

$$= 0 + \frac{1}{2}\int_{-\infty}^\infty e^{-\frac{t^2}{2}}\,dt = \frac{1}{2}\frac{1}{\sqrt{a}}\int_{-\infty}^\infty e^{-\frac{u^2}{2}}\,du = \sqrt{\frac{\pi}{2a}}. \tag{9}$$

   A more direct approach is to use the area rule for expectation:

$$E[T] = \int_0^\infty (1 - F_T(t))\,dt = \int_0^\infty e^{-\frac{at^2}{2}}\,dt = \frac{1}{2}\frac{1}{\sqrt{a}}\int_{-\infty}^\infty e^{-\frac{u^2}{2}}\,du = \sqrt{\frac{\pi}{2a}}. \tag{10}$$