ECE 313: Final Exam

Monday, May 6th, 2019
7 - 10 p.m.

1. [**8 points**] There are three distinct pairs of socks in a drawer. Three people, one by one, randomly select two socks from the drawer, without replacement. What is the probability that each person gets a pair of socks?

   Solution: Method 1: Using counting argument. The probability is equal to the number of ways of ordering three pair of socks, with each pair being together, over the total number of ways of ordering six socks. Hence

   $$\frac{3!(2^3)}{6!} = \frac{2^3}{6 \cdot 5 \cdot 4} = \frac{1}{15}.$$

   Method 2: Using conditional probability argument. Let 'A' be the event that the first person gets a pair of socks. Let 'B' be the event that the second person gets a pair of socks. Let 'C' be the event that the third person gets a pair of socks.

   $$P(ABC) = P(A)P(B|A)P(C|AB) = \frac{1}{5} \cdot \frac{1}{3} \cdot 1 = \frac{1}{15}.$$

2. [**14+5 points**] Let $X_1, X_2, \ldots, X_n$ be independent observations drawn from a Poisson distribution with unknown parameter λ.

   (a) Show that $\hat{\lambda}_{ML}$, the Maximum Likelihood estimator of λ, is the sample mean, i.e. $\hat{\lambda}_{ML} = \frac{1}{n} \sum_{i=1}^{n} X_i$. Is the ML estimator unbiased?

   Solution: We treat $X_1, X_2, \ldots, X_n$ as a set of realized observations. To find the ML estimator of λ, we maximize the corresponding likelihood function with respect to λ:

   $$L(\lambda) = \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{X_i}}{X_i!} = e^{-n\lambda} \frac{\lambda^{X_1 + X_2 + \cdots + X_n}}{X_1! X_2! \cdots X_n!}$$

   Taking the logarithm, we obtain:

   $$\ln(L(\lambda)) = -n\lambda + (X_1 + X_2 + \cdots + X_n) \ln(\lambda) - \ln(X_1! X_2! \cdots X_n!).$$

   Setting the derivative with respect to λ to zero and solving for λ, we obtain:

   $$\hat{\lambda}_{ML} = \frac{X_1 + X_2 + \cdots + X_n}{n} = \bar{X}.$$

   It can be checked that the corresponding second derivative is negative. We now observe that

   $$E[\hat{\lambda}_{ML}] = \frac{1}{n} \sum_{i=1}^{n} E[X_i] = \frac{n\lambda}{n} = \lambda.$$

   Hence, the ML estimator is unbiased.

(b) For the ML estimator $\hat{\lambda}_{ML}$ in part (a), find an upper bound of $P(|\hat{\lambda}_{ML} - \lambda| \geq \delta)$ using Chebyshev's inequality and show that $\lim_{n\to\infty} P(|\hat{\lambda}_{ML} - \lambda| \geq \delta) = 0$.

Solution: Since $E[\hat{\lambda}_{ML}] = \lambda$ and $\text{Var}(\hat{\lambda}_{ML}) = \text{Var}(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\sum_{i=1}^{n} \text{Var}(X_i) = \frac{\lambda}{n}$, the application of Chebyshev's inequality gives:

$$P(|\hat{\lambda}_{ML} - \lambda| \geq \delta) \leq \frac{\text{Var}(\hat{\lambda}_{ML})}{\delta^2} = \frac{\lambda}{n\delta^2} \xrightarrow[n\to\infty]{} 0.$$

3. **[8+4 points]** Suppose X and Y are independent random variables with joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} 2e^{-u}e^{-2v} & \text{if } u \geq 0, v \geq 0 \\ 0 & \text{else.} \end{cases}$$

(a) Find the joint pdf of $S = X + Y$ and $W = Y - X$.
Solution: Note that

$$\begin{bmatrix} S \\ W \end{bmatrix} = A \begin{bmatrix} X \\ Y \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Thus (S, W) are obtained from a linear scaling of (X, Y).

$$\det(A) = 2, \quad \text{and } A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Furthermore
$$A^{-1}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{\alpha-\beta}{2} \\ \frac{\alpha+\beta}{2} \end{bmatrix}.$$

Thus

$$f_{S,W}(\alpha, \beta) = \frac{1}{2} f_{X,Y}\left(\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}\right)$$

$$= \begin{cases} e^{-\frac{\alpha-\beta}{2}} e^{-(\alpha+\beta)} = e^{-\frac{3\alpha}{2}} e^{-\frac{\beta}{2}} & \text{if } \alpha \geq \beta, \alpha \geq -\beta \\ 0 & \text{else.} \end{cases}$$

(b) Are S and W independent? Explain.
Solution: It is easily seen that the support of the joint pdf $f_{S,W}$, which is defined by region where $\alpha > \max\{\beta, -\beta\}$ is not a product set. Therefore S and W are not independent.

4. **[7+5 points]** Consider hypotheses H_0 and H_1 about a random variable Y. Under H_0, Y has a geometric distribution with $p = 0.1$. Under H_1, Y has a geometric distribution with $p = 0.2$.

(a) Find the ML decision rule in terms of Y. You might need the following: $\frac{\ln 2}{\ln \frac{9}{8}} = 5.88$, $\frac{1}{\ln \frac{9}{8}} = 8.49$.
Solution:

$$\Lambda(k) = \frac{(0.8)^{k-1}0.2}{(0.9)^{k-1}0.1} \geq 1,$$

we have $k \leq \frac{\ln 2}{\ln \frac{9}{8}} + 1 = 6.88$. Hence we declare H_1 if $k \leq 6$ and declare H_0 if $k \geq 7$.

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(b) Find p_{miss} for the ML rule in part (a). Express your answer in the form of a^b, where a is a real number and b is an integer.

Solution:

$$p_{\text{miss}} = P(\text{declare } H_0|H_1) = \sum_{k=7}^{\infty}(0.8)^{k-1}0.2 = (0.8)^6.$$

5. **[4+4+9 points]** Let X be a geometric random variable with $p = 0.2$.

(a) Find $E[X]$.

Solution: $E[X] = \frac{1}{0.2} = 5$.

(b) Find $E[X|X > 2]$.

Solution: $E[X|X > 2] = \frac{1}{0.2} + 2 = 7$.

(c) Find $E[X^2|X > 2]$.

Solution:

$$
\begin{aligned}
E[X^2|X > 2] &= E[(X-2)^2 + 4X - 4|X > 2] \\
&= E[(X-2)^2|X > 2] + E[4X|X > 2] - 4 \\
&= Var(X) + (E[X])^2 + 4E[X|X > 2] - 4 \\
&= \frac{0.8}{0.04} + 25 + 28 - 4 \\
&= 69
\end{aligned}
$$

6. **[9+5+9 points]** Bob flips two fair coins. Let X be the number of heads that are showing. Mary now draws X cards *with replacement* from a fair deck of 52 cards. Let Y be the number of clubs that she draws. (There are four suits in a deck of cards, each suit having 13 cards, and club is one of the four suits.)

(a) Find the probability that Mary draws exactly one club.

Solution: From the law of total probability:

$$P\{Y = 1\} = P\{Y = 1|X = 0\}P\{X = 0\} + P\{Y = 1|X = 1\}P\{X = 1\}$$
$$+ P\{Y = 1|X = 2\}P\{X = 2\}$$

since $Y = 1$ is not possible if $X = 0$,

$$P\{Y = 1\} = P\{Y = 1|X = 1\}P\{X = 1\} + P\{Y = 1|X = 2\}P\{X = 2\}$$
$$= \frac{1}{4} \times \frac{1}{2} + \binom{2}{1} \times (\frac{1}{4}) \times (\frac{3}{4}) \times (\frac{1}{2})^2 = \frac{1}{8} + \frac{3}{32} = \frac{7}{32}$$

(b) Suppose Mary draws exactly one club. Find the probability that Bob had tossed two heads.

Solution: This is same as calculating $P\{X = 2|Y = 1\}$. Therefore,

$$
\begin{aligned}
P\{X = 2|Y = 1\} &= \frac{P\{X = 2, Y = 1\}}{P\{Y = 1\}} = \frac{P\{Y = 1|X = 2\}P\{X = 2\}}{P\{Y = 1\}} \\
&= \frac{\frac{3}{32}}{\frac{7}{32}} = \frac{3}{7}
\end{aligned}
$$

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(c) If Bob gets 2 points for each head and Mary gets 4 points for each club drawn, and they split the total number of points equally between themselves, find the expected value of the points they each receive.

Solution: Each receives $W = 0.5 \times (2X + 4Y)$ points. Thus, $E[W] = E[X] + 2E[Y]$. Here $E[X] = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1$. Since Y depends on X, we calculate $E[Y]$ as follows:

$$E[Y] = E[Y|X = 0]P\{X = 0\} + E[Y|X = 1]P\{X = 1\} + E[Y|X = 2]P\{X = 2\}$$
$$= 0 \times \frac{1}{4} + (0 \times \frac{3}{4} + 1 \times \frac{1}{4}) \times \frac{1}{2} + (0 \times \frac{9}{16} + 1 \times \frac{6}{16} + 2 \times \frac{1}{16})\frac{1}{4}$$
$$= \frac{1}{4}$$

Hence, $E[W] = 1 + 2 \times \frac{1}{4} = \frac{3}{2}$.

7. **[7+16+7 points]** Suppose $X \sim N(1, 1)$ and $Y \sim N(1, 4)$ are independent Gaussian random variables. Define the random variables $Z = 2X + Y$ and $W = X - Y$.

(a) Find the unconstrained MMSE estimator of Y given X, and the resulting MSE.
    Solution: Since X and Y are independent,

$$E[Y|X = u] = \int_{-\infty}^{\infty} v f_{Y|X}(v|u)dv = \int_{-\infty}^{\infty} v f_Y(v)dv = E[Y] = 1$$

i.e., the unconstrained MMSE estimator is a constant estimator. Hence, the minimum MSE is $\mathrm{Var}(Y) = 4$.

(b) Find the unconstrained MMSE estimator of Z given W, and the resulting MSE.

Solution: Since Z and W are jointly Gaussian RVs, $E[Z|W] = \hat{E}[Z|W]$, i.e., the unconstrained MMSE estimator is the same as the MMSE linear estimator. Therefore,

$$E[Y|X] = \hat{E}[Z|W] = \mu_Z + \frac{Cov(Z, W)}{Var(W)}(W - \mu_W)$$
$$MSE = \sigma_Z^2(1 - \rho_{z,w}^2)$$

We calculate the mean, variance and covariance,

$$\mu_Z = 2\mu_X + \mu_Y = 2 \times 1 + 1 = 3; \quad \mu_W = \mu_X - \mu_Y = 0$$
$$Var(W) = Cov(X - Y, X - Y) = Var(X) - 2Cov(X, Y) + Var(Y) = 1 - 0 + 4 = 5$$
$$Var(Z) = Cov(2X + Y, 2X + Y) = 4Var(X) + 4Cov(X, Y) + Var(Y)$$
$$= 4 - 0 + 4 = 8$$
$$Cov(Z, W) = Cov(2X + Y, X - Y) = 2Var(X) + Cov(X, Y) - Var(Y)$$
$$= 2 \times 1 + 0 - 4 = -2$$
$$\rho_{Z,W} = \frac{Cov(Z, W)}{\sigma_W \sigma_Z} = \frac{-2}{\sqrt{8 \times 5}} = \frac{-1}{\sqrt{10}}$$

and using these to compute:

$$E[Z|W] = 3 + \frac{-1}{\sqrt{10}} \times \sqrt{\frac{8}{5}} \times (W) = 3 - \frac{2}{5}W$$
$$MSE = \sigma_Z^2(1 - \rho_{Z,W}^2) = 8 \times (1 - (\frac{-1}{\sqrt{10}})^2) = \frac{36}{5}$$

(c) If instead $W = X - aY$ for some real a and $E[Z|W] = E[Z]$, find a.
Solution: If $E[Z|W] = E[Z]$ this implies that Z and W are independent and hence uncorrelated, i.e., $Cov(Z, W) = 0$. Therefore,

$$Cov(Z, W) = Cov(2X + Y, X - aY) = 2Var(X) - 2aCov(X, Y) + Cov(X, Y) - aVar(Y)$$

$$= 2 - 4a = 0 \implies a = \frac{1}{2}$$

8. **[7+5 points]** The probability that a circuit board coming off an assembly line needs rework is 0.1. Suppose that 10 boards are tested and all boards are independent of each other, and let X be the number of circuit boards that need rework.

(a) Find $E[X^2]$
Solution: $X \sim binom(10, 0.1)$.
$E[X] = 10 \times 0.1 = 1$.
$Var(X) = 10 \times 0.1 \times 0.9 = 0.9$
$E[X^2] = Var(X) + (E[X])^2 = 1.9$

(b) What is the probability that exactly 8 boards need rework?
Solution: $P\{X = 8\} = \binom{10}{8} \times 0.1^8 \times 0.9^2 = \binom{10}{2} \times 0.1^8 \times 0.9^2 = \frac{729}{2 \times 10^9}$.

9. **[8+9 points]** Suppose X and Y have joint pdf
$$f_{X,Y}(u, v) = \begin{cases} 2e^{-2v} & 0 \le u \le 1, v \ge 0 \\ 0 & \text{else} \end{cases}$$

(a) Find $P\{X \ge Y\}$
Solution: $P\{X \ge Y\} = \int_0^1 \int_0^u 2e^{-2v} dvdu = \int_0^1 (1 - e^{-2u})du = \frac{1 + e^{-2}}{2}$

(b) Find $P\{Xe^Y \le 1\}$
Solution: $P\{Xe^Y \le 1\} = \int_0^1 \int_0^{-lnu} ae^{-2v} dvdu = \int_0^1 (1 - u^2)du = \frac{2}{3}$

10. **[10+10 points]** Suppose X and Y are jointly Gaussian with the following parameters: $\mu_x = 0$, $\mu_y = 0$, $\sigma_x^2 = 1$, $\sigma_y^2 = 2^2$, $\rho = 1/8$.

(a) Find $P\{2X + Y \ge 3\}$. Express your answer using the Q function.
Solution: $2X + Y$ is Gaussian
$E[2X + Y] = 0$
$Var(2X + Y) = COV(2X + Y, 2X + Y) = 4Var(X) + 4COV(X, Y) + Var(Y)$
$= 4 + 4 \times \frac{1}{8} \times 2 + 4 = 9$
$P\{2X + Y \ge 3\} = P\{\frac{2X + Y}{3} \ge \frac{3}{3}\} = Q(1)$

(b) Find $E[Y^2|X = 2]$
Solution: $E[Y|X = 2] = \mu_x + \sigma_Y \rho \left(\frac{2 - \mu_x}{\sigma_X}\right) = 1/2$
$Var(Y|X = 2) = \sigma_Y^2(1 - \rho^2) = 63/16$
$E[Y^2|X = 2] = Var(Y|X = 2) + (E[Y|X = 2])^2 = 63/16 + 1/4 = 67/16$.

11. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Consider a Poisson process of rate 1. Let T_1 be the time of the first count and T_2 be the time of the second count.

TRUE   FALSE

☐   ☐   The number of arrivals between T_1 and T_2 is a Poisson random variable.

☐   ☐ T_1 has the exponential distribution with parameter 1

☐   ☐ $T_2 - T_1$ has the exponential distribution with parameter 2

Solution: False,True,False

(b) Suppose function $F_X(u)$ is the CDF of random variale X.

TRUE   FALSE

☐   ☐ $F_X(c) = F_X(c-)$ must always hold for all c.

☐   ☐ $F_X(u)$ is always monotonically increasing.

Solution: False, False

(c) Suppose X and Y are two random variables.

TRUE   FALSE

☐   ☐   If $\text{Cov}(X, Y) = 0$ then $\hat{E}[Y|X] = E[Y]$.

☐   ☐   If $\text{Cov}(X, Y^2) = 1$, X and Y are dependent.

Solution: True, True

(d) Consider the binary hypothesis problem. Let the probability of false alarm and missed detection for the ML rule be denoted by p_f^{ML} and p_m^{ML}, respectively. Similarly, let the probability of false alarm and missed detection for the MAP rule be denoted by p_f^{MAP} and p_m^{MAP}, respectively.

TRUE   FALSE

☐   ☐ $p_f^{MAP} + p_m^{MAP} = 1$.

☐   ☐ $\pi_0 p_f^{MAP} + \pi_1 p_m^{MAP} \geq \pi_0 p_f^{ML} + \pi_1 p_m^{ML}$.

☐   ☐   If $\pi_1 = 0.5$ then $p_f^{ML} = p_f^{MAP}$.

Solution: False, False, True