ECE 313: Final Exam

Monday, May 6th, 2019
7 - 10 p.m.

1. **[8 points]** There are three distinct pairs of socks in a drawer. Three people, one by one, randomly select two socks from the drawer, without replacement. What is the probability that each person gets a pair of socks?

   Solution: Method 1: Using counting argument. The probability is equal to the number of ways of ordering three pair of socks, with each pair being together, over the total number of ways of ordering six socks. Hence

   $$\frac{3!(2^3)}{6!} = \frac{2^3}{6 \cdot 5 \cdot 4} = \frac{1}{15}.$$

   Method 2: Using conditional probability argument. Let 'A' be the event that the first person gets a pair of socks. Let 'B' be the event that the second person gets a pair of socks. Let 'C' be the event that the third person gets a pair of socks.

   $$P(ABC) = P(A)P(B|A)P(C|AB) = \frac{1}{5} \cdot \frac{1}{3} \cdot 1 = \frac{1}{15}.$$

2. **[8+4 points]** Suppose X and Y are independent random variables with joint pdf:

   $$f_{X,Y}(u,v) = \begin{cases} 2e^{-u}e^{-2v} & \text{if } u \geq 0, v \geq 0 \\ 0 & \text{else.} \end{cases}$$

   (a) Find the joint pdf of $S = X + Y$ and $W = Y - X$.
   Solution: Note that

   $$\begin{bmatrix} S \\ W \end{bmatrix} = A \begin{bmatrix} X \\ Y \end{bmatrix}, \text{ where } A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

   Thus (S, W) are obtained from a linear scaling of (X, Y).

   $$\det(A) = 2, \text{ and } A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

   Furthermore

   $$A^{-1}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{\alpha-\beta}{2} \\ \frac{\alpha+\beta}{2} \end{bmatrix}.$$

   Thus

   $$\begin{aligned} f_{S,W}(\alpha,\beta) &= \frac{1}{2} f_{X,Y}\left(\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}\right) \\ &= \begin{cases} e^{-\frac{\alpha-\beta}{2}} e^{-(\alpha+\beta)} = e^{-\frac{3\alpha}{2}} e^{-\frac{\beta}{2}} & \text{if } \alpha \geq \beta, \alpha \geq -\beta \\ 0 & \text{else.} \end{cases} \end{aligned}$$

(b) Are S and W independent? Explain.
Solution: It is easily seen that the support of the joint pdf $f_{S,W}$, which is defined by region where $\alpha > \max\{\beta, -\beta\}$ is not a product set. Therefore S and W are not independent.

3. **[9+5+9 points]** Bob flips two fair coins. Let X be the number of heads that are showing. Mary now draws X cards *with replacement* from a fair deck of 52 cards. Let Y be the number of clubs that she draws. (There are four suits in a deck of cards, each suit having 13 cards, and club is one of the four suits.)

(a) Find the probability that Mary draws exactly one club.
Solution: From the law of total probability:

$$\begin{aligned} P\{Y=1\} &= P\{Y=1|X=0\}P\{X=0\} + P\{Y=1|X=1\}P\{X=1\} \\ &\quad + P\{Y=1|X=2\}P\{X=2\} \end{aligned}$$

since $Y=1$ is not possible if $X=0$,

$$\begin{aligned} P\{Y=1\} &= P\{Y=1|X=1\}P\{X=1\} + P\{Y=1|X=2\}P\{X=2\} \\ &= \frac{1}{4} \times \frac{1}{2} + \binom{2}{1} \times (\frac{1}{4}) \times (\frac{3}{4}) \times (\frac{1}{2})^2 = \frac{1}{8} + \frac{3}{32} = \frac{7}{32} \end{aligned}$$

(b) Suppose Mary draws exactly one club. Find the probability that Bob had tossed two heads.
Solution: This is same as calculating $P\{X=2|Y=1\}$. Therefore,

$$\begin{aligned} P\{X=2|Y=1\} &= \frac{P\{X=2, Y=1\}}{P\{Y=1\}} = \frac{P\{Y=1|X=2\}P\{X=2\}}{P\{Y=1\}} \\ &= \frac{\frac{3}{32}}{\frac{7}{32}} = \frac{3}{7} \end{aligned}$$

(c) If Bob gets 2 points for each head and Mary gets 4 points for each club drawn, and they split the total number of points equally between themselves, find the expected value of the points they each receive.
Solution: Each receives $W = 0.5 \times (2X + 4Y)$ points. Thus, $E[W] = E[X] + 2E[Y]$. Here $E[X] = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1$. Since Y depends on X, we calculate $E[Y]$ as follows:

$$\begin{aligned} E[Y] &= E[Y|X=0]P\{X=0\} + E[Y|X=1]P\{X=1\} + E[Y|X=2]P\{X=2\} \\ &= 0 \times \frac{1}{4} + (0 \times \frac{3}{4} + 1 \times \frac{1}{4}) \times \frac{1}{2} + (0 \times \frac{9}{16} + 1 \times \frac{6}{16} + 2 \times \frac{1}{16})\frac{1}{4} \\ &= \frac{1}{4} \end{aligned}$$

Hence, $E[W] = 1 + 2 \times \frac{1}{4} = \frac{3}{2}$.

4. **[7+7+7+4 points]** Consider the binary hypothesis problem in which the pdfs of X under hypotheses H_0 and H_1 are given by

$$f_0(u) = \begin{cases} 2u & \text{if } 0 \le u \le 1 \\ 0 & \text{else} \end{cases}$$

$$f_1(u) = \begin{cases} 3u^2 & \text{if } 0 \le u < 1 \\ 0 & \text{else} \end{cases}$$

with priors $\pi_0 = 3\pi_1$.

(a) Find the ML decision rule.
Solution: The likelihood function is given by:

$$\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \frac{3u^2}{2u} = \frac{3}{2}u$$

for $0 \leq u \leq 1$. Thus, the ML decision rule is:

$$\begin{aligned} X > \frac{2}{3} : &\quad \text{declare } H_1 \text{ is true} \\ X < \frac{2}{3} : &\quad \text{declare } H_0 \text{ is true} \end{aligned}$$

(b) Calculate p_{miss} and $p_{\text{false-alarm}}$ for the ML decision rule.
Solution:

$$\begin{aligned} p_{\text{miss}} &= P\{\text{declare } H_0|H_1\} = P\{\{0 \leq X < \frac{2}{3}\}|H_1\} = \int_0^{2/3} 3u^2 du = \frac{8}{27} \\ p_{\text{false-alarm}} &= P\{\text{declare } H_1|H_0\} = P\{\{\frac{2}{3} < X \leq 1\}|H_0\} = \int_{2/3}^{1} 2u du = \frac{5}{9} \end{aligned}$$

(c) The MAP decision rule turns out to always declare H_0 to be true. Find the minimum value of the ratio $\frac{\pi_0}{\pi_1}$ of priors for this to occur.

Solution: Since the likelihood function $\Lambda(u) = \frac{3}{2}u$, the MAP decision rule is to declare H_1 if $\Lambda(u) > \frac{\pi_0}{\pi_1}$ or $u > \frac{\pi_0}{\pi_1}\frac{2}{3}$. Since both $f_0(u)$ and $f_1(u)$ have support $0 \leq u \leq 1$, we just need the R.H.S. of this inequality to be equal to or greater than 1 in order for the MAP rule to always choose H_0. Thus, the minimum value of the ratio of priors $\frac{\pi_0}{\pi_1} = \frac{3}{2}$ for this to happen.

(d) Calculate p_{miss} and $p_{\text{false-alarm}}$ for the MAP decision rule which always declares H_0 to be true.
Solution:

$$\begin{aligned} p_{\text{miss}} &= P\{\text{declare } H_0|H_1\} = 1 \\ p_{\text{false-alarm}} &= P\{\text{declare } H_1|H_0\} = 0 \end{aligned}$$

5. **[7+16+7 points]** Suppose $X \sim N(1,1)$ and $Y \sim N(1,4)$ are independent Gaussian random variables. Define the random variables $Z = 2X + Y$ and $W = X - Y$.

(a) Find the unconstrained MMSE estimator of Y given X, and the resulting MSE.
Solution: Since X and Y are independent,

$$E[Y|X = u] = \int_{-\infty}^{\infty} v f_{Y|X}(v|u) dv = \int_{-\infty}^{\infty} v f_Y(v) dv = E[Y] = 1$$

i.e., the unconstrained MMSE estimator is a constant estimator. Hence, the minimum MSE is $\text{Var}(Y) = 4$.

(b) Find the unconstrained MMSE estimator of Z given W, and the resulting MSE.

Solution: Since Z and W are jointly Gaussian RVs, $E[Z|W] = \hat{E}[Z|W]$, i.e., the unconstrained MMSE estimator is the same as the MMSE linear estimator. Therefore,

$$\begin{aligned} E[Y|X] = \hat{E}[Z|W] &= \mu_Z + \frac{Cov(Z,W)}{Var(W)}(W - \mu_W) \\ MSE &= \sigma_Z^2(1 - \rho_{z,w}^2) \end{aligned}$$

We calculate the mean, variance and covariance,

$$\begin{aligned} \mu_Z &= 2\mu_X + \mu_Y = 2 \times 1 + 1 = 3; \quad \mu_W = \mu_X - \mu_Y = 0 \\ Var(W) &= Cov(X - Y, X - Y) = Var(X) - 2Cov(X,Y) + Var(Y) = 1 - 0 + 4 = 5 \\ Var(Z) &= Cov(2X + Y, 2X + Y) = 4Var(X) + 4Cov(X,Y) + Var(Y) \\ &= 4 - 0 + 4 = 8 \\ Cov(Z,W) &= Cov(2X + Y, X - Y) = 2Var(X) + Cov(X,Y) - Var(Y) \\ &= 2 \times 1 + 0 - 4 = -2 \\ \rho_{Z,W} &= \frac{Cov(Z,W)}{\sigma_W \sigma_Z} = \frac{-2}{\sqrt{8 \times 5}} = \frac{-1}{\sqrt{10}} \end{aligned}$$

and using these to compute:

$$\begin{aligned} E[Z|W] &= 3 + \frac{-1}{\sqrt{10}} \times \sqrt{\frac{8}{5}} \times (W) = 3 - \frac{2}{5}W \\ MSE &= \sigma_Z^2(1 - \rho_{Z,W}^2) = 8 \times (1 - (\frac{-1}{\sqrt{10}})^2) = \frac{36}{5} \end{aligned}$$

(c) If instead $W = X - aY$ for some real a and $E[Z|W] = E[Z]$, find a.

Solution: If $E[Z|W] = E[Z]$ this implies that Z and W are independent and hence uncorrelated, i.e., $Cov(Z,W) = 0$. Therefore,

$$\begin{aligned} Cov(Z,W) &= Cov(2X + Y, X - aY) = 2Var(X) - 2aCov(X,Y) + Cov(X,Y) - aVar(Y) \\ &= 2 - 4a = 0 \implies a = \frac{1}{2} \end{aligned}$$

6. **[7 points]** Suppose X is described by the pdf:

$$f_X(u) = \begin{cases} \frac{u}{\theta} e^{-\frac{u^2}{2\theta}} & \text{if } u \geq 0 \\ 0 & \text{else} \end{cases}$$

It is observed that $X = 10$. Find the maximum likelihood estimate $\hat{\theta}_{ML}$ of θ.

Solution: The likelihood of $X = 10$ is $f_X(10) = \frac{10}{\theta} e^{-\frac{100}{2\theta}}$. To maximize it, take log and then derivative w.r.t. θ as follows:

$$\begin{aligned} \ln(f_X(10)) &= \ln(10) - \ln(\theta) - \frac{50}{\theta} \\ \frac{\partial \ln(f_X(10))}{\partial \theta} &= -\frac{1}{\theta} + \frac{50}{\theta^2} \end{aligned}$$

This derivative is zero at $\theta = 50$, > 0 for $\theta < 50$, < 0 for $\theta > 50$, hence $\hat{\theta}_{ML} = 50$.

7. **[6+6 points]** Suppose the life time of Tom's laptop has an exponential distribution with a mean life of 1000 days.

   (a) What is the probability that the life time of Tom's laptop is longer than 400 days? Express your final answer in the form of e^a, and give the numerical value of a.
   Solution: $f_X(u) = \begin{cases} \frac{1}{1000}exp(-\frac{1}{1000}u), u \geq 0, \\ 0, u < 0 \end{cases}$
   $P\{X > 400\} = \int_{400}^{\infty} f_X(u)du = \int_{400}^{\infty} \frac{1}{1000}exp(-\frac{1}{1000}u)du = e^{-0.4}$
   Alternatively: $P\{x > 400\} = e^{-\lambda\times 400} = e^{-\frac{1}{1000}\times 400} = e^{-0.4}$

   (b) Suppose that Tom has used his laptop for 400 days, what is the probability that it will last for another 2000 days? Express your final answer in the form of e^b, and give the numerical value of b.
   Solution: Using the memoryless property:
   $P\{X > 2400|X > 400\} = P\{X > 2000\} = e^{-2000/1000} = e^{-2}$

8. **[8+9 points]** Suppose X and Y have joint pdf
   $f_{X,Y}(u,v) = \begin{cases} 2e^{-2v} & 0 \leq u \leq 1, v \geq 0 \\ 0 & \text{else} \end{cases}$

   (a) Find $P\{X \geq Y\}$
   Solution: $P\{X \geq Y\} = \int_0^1 \int_0^u 2e^{-2v}dvdu = \int_0^1 (1 - e^{-2u})du = \frac{1+e^{-2}}{2}$

   (b) Find $P\{Xe^Y \leq 1\}$
   Solution: $P\{Xe^Y \leq 1\} = \int_0^1 \int_0^{-lnu} ae^{-2v}dvdu = \int_0^1 (1 - u^2)du = \frac{2}{3}$

9. **[8+8 points]** Suppose $X_1, X_2, ..., X_n$ are independent, identically distributed random variables, each with mean $\mu = 2$ and variance $\sigma^2 = 4$. Let $S_n = X_1 + X_2 + ... + X_n$. Determine a condition on n so the probability the sample average $\frac{S_n}{n}$ is within 1% of the mean ($\mu = 2$), is greater than 0.95.

   (a) Solve the problem using the form of the law of large numbers based on the Chebychev inequality.
   Solution: From LLN, we have,
   $P\{\left|\frac{S_n}{n} - \mu\right| \geq 0.01\mu\} \leq \frac{\sigma^2}{n(0.01\mu)^2}$
   $\Rightarrow P\{\left|\frac{S_n}{n} - \mu\right| < 0.01\mu\} \geq 1 - \frac{\sigma^2}{n(0.01\mu)^2}$
   $\Rightarrow 1 - \frac{\sigma^2}{n(0.01\mu)^2} \geq 0.95$
   $\Rightarrow \frac{\sigma^2}{n(0.01\mu)^2} \leq 0.05$
   $\Rightarrow n \geq \frac{\sigma^2}{0.05(0.01\mu)^2}$
   $\Rightarrow n \geq 5 \times 10^6$

   (b) Solve the problem using the Gaussian approximation for S_n according to the central limit theorem. The following Q-function table is provided in case needed:
   $Q(1.28) = 0.1, Q(1.65) = 0.05, Q(1.96) = 0.025, Q(2.33) = 0.01, Q(2.58) = 0.005.$
   Solution: $E[S_n] = 2n$, $\text{Var}(S_n) = 4n$, and the standard deviation of S_n is $2\sqrt{n}$.
   From CLT, we have,
   $P\{|S_n - 2n| \leq 0.02n\} = P\{\frac{|S_n-2n|}{2\sqrt{n}} \leq \frac{0.02n}{2\sqrt{n}}\} \approx 1 - 2Q\left(\frac{0.02n}{2\sqrt{n}}\right)$
   $\Rightarrow 1 - 2Q\left(\frac{0.02n}{2\sqrt{n}}\right) \geq 0.95$
   $\Rightarrow Q\left(\frac{0.02n}{2\sqrt{n}}\right) \leq 0.025$
   $\Rightarrow \frac{0.02n}{2\sqrt{n}} \geq 1.96$
   $\Rightarrow n \geq 196^2 = 38,416$

10. **[10+10 points]** Suppose X and Y are jointly Gaussian with the following parameters: $\mu_x = 0,\ \mu_y = 0,\ \sigma_x^2 = 1,\ \sigma_y^2 = 2^2,\ \rho = 1/8.$

   (a) Find $P\{2X + Y \geq 3\}$. Express your answer using the Q function.
   Solution: $2X + Y$ is Gaussian
   $E[2X + Y] = 0$
   $\mathrm{Var}(2X + Y) = \mathrm{COV}(2X + Y, 2X + Y) = 4\mathrm{Var}(X) + 4\mathrm{COV}(X, Y) + \mathrm{Var}(Y)$
   $= 4 + 4 \times \frac{1}{8} \times 2 + 4 = 9$
   $P\{2X + Y \geq 3\} = P\{\frac{2X+Y}{3} \geq \frac{3}{3}\} = Q(1)$

   (b) Find $E[Y^2|X = 2]$
   Solution: $E[Y|X = 2] = \mu_x + \sigma_Y \rho \left(\frac{2-\mu_x}{\sigma_X}\right) = 1/2$
   $\mathrm{Var}(Y|X = 2) = \sigma_Y^2(1 - \rho^2) = 63/16$
   $E[Y^2|X = 2] = \mathrm{Var}(Y|X = 2) + (E[Y|X = 2])^2 = 63/16 + 1/4 = 67/16.$

11. **[30 points]** (3 points per answer)
   In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

   (a) Consider a Poisson process of rate 1. Let T_1 be the time of the first count and T_2 be the time of the second count.

TRUE	FALSE	
☐	☐	The number of arrivals between T_1 and T_2 is a Poisson random variable.
☐	☐	$T_2 - T_1$ has the exponential distribution with parameter 2

   Solution: False, False

   (b) Suppose function $F_X(u)$ is the CDF of random variale X.

TRUE	FALSE	
☐	☐	$F_X(c) = F_X(c-)$ must always hold for all c.
☐	☐	$F_X(u)$ is always monotonically increasing.

   Solution: False, False

   (c) Suppose X and Y are two random variables with a joint pdf $f_{X,Y}(u, v)$.

TRUE	FALSE	
☐	☐	If $Cov(X, Y) = 0$ then $\hat{E}[Y|X] = E[Y]$.
☐	☐	If X and Y are dependent then Y is a function of X.
☐	☐	If $f_{X,Y}(u, v) = f_X(u)f_Y(v)$ then $E[Y|X] = E[Y]$.

   Solution: True, False, True

   (d) Consider the binary hypothesis problem. Let the probability of false alarm and missed detection for the ML rule be denoted by p_f^{ML} and p_m^{ML}, respectively. Similarly, let the probability of false alarm and missed detection for the MAP rule be denoted by p_f^{MAP} and p_m^{MAP}, respectively.

TRUE	FALSE	
☐	☐	$p_f^{MAP} + p_m^{MAP} = 1$.
☐	☐	$\pi_0 p_f^{MAP} + \pi_1 p_m^{MAP} \geq \pi_0 p_f^{ML} + \pi_1 p_m^{ML}$.
☐	☐	If $\pi_1 = 0.5$ then $p_f^{ML} = p_f^{MAP}$.

Solution: False, False, True