University of Illinois Spring 2019

ECE 313: Hour Exam II

Wednesday, April 10, 2019
8:45 p.m. — 10:00 p.m.

1. **[5+5+11+5 points]** Consider a Poisson process with rate 1. Define X to be the total number of counts during $[0,3]$, and Y to be the number of counts during $[0,1]$. Let T be the time of the first count. Express your answers to the following questions in terms of e.

   (a) Find $P(X=2)$.
   Solution: $E(X) = 1 \times 3 = 3$, so $X \sim$Poisson(3).

   $$P(X=2) = \frac{e^{-3}3^2}{2!} = \frac{9}{2}e^{-3}.$$

   (b) Find $P(T>2)$.
   Solution: Since $T \sim$Exp(1), $P(T>2) = e^{-1\times 2} = e^{-2}$.

   (c) Find $P(X=2|Y=1)$.
   Solution: Let Z be the number of counts during $[1,3]$, so $Z \sim$Poisson(2) . And Y and Z are independent since the two intervals do not overlap.

   $$\begin{aligned} P(X=2|Y=1) &= \frac{P(X=2,Y=1)}{P(Y=1)} = \frac{P(Y=1,Z=1)}{P(Y=1)} \\ &= \frac{P(Y=1)P(Z=1)}{P(Y=1)} = P(Z=1) = \frac{e^{-2}2^1}{1!} = 2e^{-2}. \end{aligned}$$

   (d) Find $E(T|T>2)$
   Solution: Since T $\sim$Exp(1), using the memoryless property of exponential random variables,

   $$E(T|T>2) = E(T) + 2 = \frac{1}{1} + 2 = 3.$$

2. **[12 points]** Let $T_1, T_2, \cdots, T_{10}$ be i.i.d. exponentially distributed random variables with parameter λ. Suppose we observe that five out of the ten random variables have values greater than 1. Find the ML estimate, $\hat{\lambda}$, of λ.

   Solution: $P(T_i > 1) = e^{-\lambda}$, $i = 1, 2, \cdots, 10$. Hence, the likelihood function

   $$L(\lambda) = \binom{10}{5}(e^{-\lambda})^5(1-e^{-\lambda})^5.$$

   Maximizing $L(\lambda)$ by taking log and differentiate:

   $$\begin{aligned} \ln(L(\lambda)) &= \ln\binom{10}{5} - 5\lambda + 5\ln(1-e^{-\lambda}) \\ \frac{d}{d\lambda}\ln(L(\lambda)) &= -5 + \frac{5e^{-\lambda}}{1-e^{-\lambda}} \end{aligned}$$

Setting $\frac{d}{d\lambda}\ln(L(\lambda))$ to 0 and solve for λ, we obtain

$$\begin{aligned} e^{-\lambda} &= 1 - e^{-\lambda} \\ e^{-\lambda} &= \frac{1}{2} \\ \lambda &= \ln 2 \end{aligned}$$

Hence, $\hat{\lambda} = \ln 2$.

3. **[8+12 points]** A random variable X has a $N(2,9)$ distribution and $Y = 3X + 2$.

   (a) Find an expression for the density function of Y, $E[Y]$ and $\text{Var}(Y)$.
   Solution: Since Y is a linearly scaled version of X, its density function is also Gaussian. A Gaussian density function is completely described by its mean and variance. Therefore, $E[Y] = 3E[X]+2 = 3\times 2+2 = 8$ and $\text{Var}(Y) = 9\text{Var}(X) = 81$ and hence Y is $N(8,81)$, i.e.,

   $$f_Y(v) = \frac{1}{\sqrt{162\pi}} e^{-\frac{(v-8)^2}{162}}$$

   Alternative approach is to use the linear scaling formula: if $Y = aX + b$ $(a > 0)$ then $f_Y(v) = (1/a)f_X(\frac{v-b}{a})$ with $a = 3$ and $b = 2$.

   (b) Find $P\{|3X+1| > 2\}$ in terms of the CDF of a standard normal ($\Phi(x)$.)
   Solution:

   $$\begin{aligned} P\{|3X+1|>2\} &= P\{\{3X+1>+2\}\cup\{3X+1<-2\}\} = P\{\{X>+\frac{1}{3}\}\cup\{X<-1\}\} \\ &= P\left\{X > +\frac{1}{3}\right\} + P\{X<-1\} = P\left\{\frac{X-2}{3} > -\frac{5}{9}\right\} + P\left\{\frac{X-2}{3} < -1\right\} \\ &= 1 - \Phi\left(-\frac{5}{9}\right) + \Phi(-1) \end{aligned}$$

4. **[12+10 points]** Consider the binary hypothesis problem in which the pdfs of X under hypotheses H_0 and H_1 are given by

   $$f_0(u) = \begin{cases} \frac{1}{2} & \text{if } 0 \le u \le 2 \\ 0 & \text{otherwise} \end{cases}$$

   $$f_1(u) = \begin{cases} u & \text{if } 0 \le u < 1 \\ 2-u & \text{if } 1 \le u \le 2 \\ 0 & \text{otherwise} \end{cases}$$

   with priors $\pi_1 = 2\pi_0$.

   (a) Write an expression for the likelihood function $\Lambda(u)$ and find the MAP rule.
   Solution:

   $$\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \begin{cases} 2u & \text{if } 0 \le u < 1 \\ 4-2u & \text{if } 1 \le u \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Since, the threshold for the MAP rule is $\frac{\pi_0}{\pi_1} = 0.5$, the likelihood ratio test is given as:

$$\Lambda(X) \begin{cases} > 0.5 & \text{declare } H_1 \text{ is true} \\ < 0.5 & \text{declare } H_0 \text{ is true} \end{cases}$$

Hence, the MAP decision rule is given by:

$$\frac{1}{4} < X < \frac{7}{4}: \quad \text{declare } H_1 \text{ is true}$$
$$0 < X < \frac{1}{4} \quad \text{or} \quad \frac{7}{4} < X < 2: \quad \text{declare } H_0 \text{ is true}$$

(b) Calculate p_{miss}, $p_{\text{false-alarm}}$, and error probability p_e.
Solution: Since $\pi_0 + \pi_1 = 1$ and $\pi_1 = 2\pi_0 \implies \pi_0 = \frac{1}{3}, \pi_1 = \frac{2}{3}$

$$p_{\text{miss}} = P\{\text{declare } H_0|H_1\} = P\{\{0 < X < \frac{1}{4}\} \cup \{\frac{7}{4} < X < 2\}|H_1\} = \frac{1}{16}$$
$$p_{\text{false-alarm}} = P\{\text{declare } H_1|H_0\} = P\{\{\frac{1}{4} < X < \frac{7}{4}\}|H_0\} = \frac{3}{4}$$
$$p_e = p_{\text{miss}} \times \pi_1 + p_{\text{false-alarm}} \times \pi_0 = \frac{1}{16} \times \frac{2}{3} + \frac{3}{4} \times \frac{1}{3} = \frac{7}{24}$$

5. **[8+8+4 points]** Let X and Y be continuous-type random variables with joint pdf $f_{X,Y}(u,v) = \begin{cases} 2, & 0 \le v \le u \le 1 \\ 0, & \text{else.} \end{cases}$
Then the marginal pdf of Y is $f_Y(v_o) = \begin{cases} 2(1-v_o), & 0 \le v_o < 1 \\ 0, & \text{else.} \end{cases}$

(a) Find $f_X(u)$, the marginal pdf of X.
Solution: $f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)dv = \begin{cases} \int_0^u 2dv = 2u, & 0 \le u \le 1 \\ 0, & \text{else.} \end{cases}$

(b) Find $f_{X|Y}(u|v_o)$, the conditional pdf of X given Y.

Solution: For $v_o < 0$ or $v_o \ge 1$, $f_{X|Y}(u|v_o)$ is undefined.
For $0 \le v_o < 1$, $f_{X|Y}(u|v_o) = \frac{f_{X,Y}(u,v_o)}{f_Y(v_o)} = \begin{cases} \frac{1}{1-v_o}, & v_o \le u \le 1 \\ 0, & \text{else.} \end{cases}$

(c) Are X and Y independent? Why?
Solution: No, X and Y are not independent. Many reasons are acceptable. Here are a few examples:
i) The support of $f_{X,Y}$ is not a product set.
ii) For $f_X \ne 0$, $f_{X|Y} = f_X$ does not always hold.
iii) $f_{X,Y} \ne f_X f_Y$.