University of Illinois Spring 2019

ECE 313: Hour Exam II

Wednesday, April 10, 2019
8:45 p.m. — 10:00 p.m.

1. **[6+6 points]** Suppose X is uniformly distributed on $[0, 2]$.

   (a) Find $E(X^2)$.
   Solution:

   $$E(X^2) = \int_0^2 \frac{1}{2}u^2 du = \left[\frac{u^3}{6}\right]_0^2 = \frac{4}{3}.$$

   (b) Find $F_X(1)$.
   Solution:

   $$F_X(1) = P(X \leq 1) = \int_0^1 \frac{1}{2} du = \frac{1}{2}.$$

2. **[10+10+8 points]** Three random variables are defined: X is exponentially distributed with parameter $\lambda = 2$, Y is Gaussian with $N(2, 9)$ and U is uniformly distributed in $[0, 1]$. Answer the following:

   (a) Find a function g such that $g(X) = U$. Assume that g is non-decreasing and g^{-1} exists. Show steps for full credit.
   Solution: Let c be the value taken by U.

   $$F_U(c) = P\{U \leq c\} = c = P\{g(X) \leq c\} = P\{X \leq g^{-1}(c)\} = F_X(g^{-1}(c))$$
   $$\implies F_X(g^{-1}(c)) = c \implies g^{-1}(c) = F_X^{-1}(c) \implies g(c) = F_X(c)$$

   Since $X \sim Exp(2)$, hence $g(X) = 1 - e^{-2X} = U$.

   (b) Find h such that $h(U) = Y$. Assume that h is non-decreasing and h^{-1} exists. Show steps for full credit.
   Solution: Let u be the value taken by U and c be the value taken by Y.

   $$F_Y(c) = \Phi(\frac{c-2}{3}) = P\{Y \leq c\} = P\{h(U) \leq c\} = P\{U \leq h^{-1}(c)\} = F_U(h^{-1}(c)) = h^{-1}(c)$$
   $$\implies h^{-1}(c) = u = \Phi(\frac{c-2}{3}) \implies h(u) = 3\Phi^{-1}(u) + 2$$

   Hence, $h(U) = 3\Phi^{-1}(U) + 2 = Y$.

   (c) Find a function k such that $k(X) = Y$. Assume that k is non-decreasing and k^{-1} exists. Show steps for full credit.
   Solution: Let u be the value taken by X and c be the value taken by Y.

   $$F_Y(c) = \Phi(\frac{c-2}{3}) = P\{Y \leq c\} = P\{X \leq k^{-1}(c)\} = F_X(k^{-1}(c)) = 1 - e^{-2k^{-1}(c)}$$
   $$\implies k^{-1}(c) = u = -\frac{1}{2}\ln(1 - \Phi(\frac{c-2}{3})) \implies k(u) = c = 3\Phi^{-1}(1 - e^{-2u}) + 2$$

   Hence, $k(X) = 3\Phi^{-1}(1 - e^{-2X}) + 2 = Y$. Alternatively, $k(X) = h(g(X))$ but you need parts a) and b) to be correct.

3. **[12+10 points]** Consider the binary hypothesis problem in which the pdfs of X under hypotheses H_0 and H_1 are given by

$$f_0(u) = \begin{cases} \frac{1}{2} & \text{if} \quad 0 \le u \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_1(u) = \begin{cases} u & \text{if} \quad 0 \le u < 1 \\ 2-u & \text{if} \quad 1 \le u \le 2 \\ 0 & \text{otherwise} \end{cases}$$

with priors $\pi_1 = 2\pi_0$.

   (a) Write an expression for the likelihood function $\Lambda(u)$ and find the MAP rule.
   Solution:

$$\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \begin{cases} 2u & \text{if} \quad 0 \le u < 1 \\ 4-2u & \text{if} \quad 1 \le u \le 2 \\ 0 & \text{otherwise} \end{cases}$$

   Since, the threshold for the MAP rule is $\frac{\pi_0}{\pi_1} = 0.5$, the likelihood ratio test is given as:

$$\Lambda(X) \begin{cases} > 0.5 & \text{declare } H_1 \text{ is true} \\ < 0.5 & \text{declare } H_0 \text{ is true} \end{cases}$$

   Hence, the MAP decision rule is given by:

$$\begin{aligned} \frac{1}{4} < X < \frac{7}{4}: & \quad \text{declare } H_1 \text{ is true} \\ 0 < X < \frac{1}{4} \quad \text{or} \quad \frac{7}{4} < X < 2: & \quad \text{declare } H_0 \text{ is true} \end{aligned}$$

   (b) Calculate p_{miss}, $p_{\text{false-alarm}}$, and error probability p_e.
   Solution: Since $\pi_0 + \pi_1 = 1$ and $\pi_1 = 2\pi_0 \implies \pi_0 = \frac{1}{3}, \pi_1 = \frac{2}{3}$

$$\begin{aligned} p_{\text{miss}} &= P\{\text{declare } H_0 | H_1\} = P\{\{0 < X < \frac{1}{4}\} \cup \{\frac{7}{4} < X < 2\} | H_1\} = \frac{1}{16} \\ p_{\text{false-alarm}} &= P\{\text{declare } H_1 | H_0\} = P\{\{\frac{1}{4} < X < \frac{7}{4}\} | H_0\} = \frac{3}{4} \\ p_e &= p_{\text{miss}} \times \pi_1 + p_{\text{false-alarm}} \times \pi_0 = \frac{1}{16} \times \frac{2}{3} + \frac{3}{4} \times \frac{1}{3} = \frac{7}{24} \end{aligned}$$

4. **[8+6+8 points]** Suppose X and Y are discrete-type random variables with the joint pmf given by
$p_{X,Y}(u,v) = \frac{u+v}{32}, u = 1,2, v = 1,2,3,4$.

   (a) Find $p_X(u)$, the marginal pmf of X.
   Solution: $p_X(1) = \frac{1+1}{32} + \frac{1+2}{32} + \frac{1+3}{32} + \frac{1+4}{32} = \frac{7}{16}$
   $p_X(2) = \frac{2+1}{32} + \frac{2+2}{32} + \frac{2+3}{32} + \frac{2+4}{32} = \frac{9}{16}$
   Alternatively, $p_X(u) = \frac{2u+5}{16}, u = 1,2$.

   (b) Find $P\{Y = 2X\}$
   Solution: $P\{Y = 2X\} = p_{X,Y}(1,2) + p_{X,Y}(2,4) = \frac{3}{32} + \frac{6}{32} = \frac{9}{32}$

(c) Find $p_{X|Y}(u|2)$, the conditional probability of X given $Y = 2$.

Solution: $p_Y(2) = \frac{1+2}{32} + \frac{2+2}{32} = \frac{7}{32}$

$p_{X|Y}(u|2) = \begin{cases} \frac{3/32}{7/32} = \frac{3}{7}, u = 1 \\ \frac{4/32}{7/32} = \frac{4}{7}, u = 2 \end{cases}$, or $p_{X|Y}(u|2) = \frac{u+2}{7}, u = 1, 2.$

5. **[6+10 points]** Suppose X and Y are independent random variables with probability density functions $f_X(u) = \begin{cases} 3u^2 & 0 \le u \le 1 \\ 0 & \text{else,} \end{cases}$ and $f_Y(v) = \begin{cases} 2v & 0 \le v \le 1 \\ 0 & \text{else,} \end{cases}$ respectively.

(a) Find the joint pdf of (X, Y).

Solution: Since X and Y are independent, the joint pdf is

$f_{X,Y}(u, v) = f_X(u)f_Y(v) = \begin{cases} 6u^2 v, & 0 \le u \le 1, \quad 0 \le v \le 1 \\ 0, & \text{else.} \end{cases}$

(b) Find $P\{X < Y\}$.

Solution:

$$
\begin{aligned}
P(X < Y) &= \int_0^1 \left(\int_0^v 6u^2 v du \right) dv = \int_0^1 v \left(\int_0^v 6u^2 du \right) dv \\
&= \int_0^1 v \left(2u^3 \big|_0^v \right) dv = \int_0^1 2v^4 dv = \frac{2}{5} v^5 \Big|_0^1 \\
&= \frac{2}{5}.
\end{aligned}
$$

Alternative solution:

$$
\begin{aligned}
P(X < Y) &= \int_0^1 \left(\int_u^1 6u^2 v dv \right) du = \int_0^1 u^2 \left(\int_u^1 6v dv \right) du \\
&= \int_0^1 u^2 \left(3v^2 \big|_u^1 \right) du = \int_0^1 \left(3u^2 - 3u^4 \right) du = \left(u^3 - \frac{3}{5} u^5 \right) \Big|_0^1 \\
&= \frac{2}{5}.
\end{aligned}
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