University of Illinois Spring 2019

ECE 313: Hour Exam I

Wednesday, February 27, 2019
8:45 p.m. — 10:00 p.m.

1. **[10+10 points]** An experiment consists of observing the content of an 8-bit register. We assume that all 256 byte values are equally likely to be observed.

   (a) Let B denote the event that the register contains 5 ONEs and 3 ZEROes. What is $P(B)$?
   Solution: Using the principle of over counting, we index each of the 5 ONEs and each of the 3 ZEROes to make them distinct. There are 8! possible bytes using the distinct ONE and ZEROs. For any one of these, there are $5! \times 3!$ bytes are in fact the same byte if the ONEs and ZEROes are indistinct. Thus, $|B| = \frac{8!}{5! \times 3!} = 56$. Thus, $P(B) = \frac{56}{256} = \frac{7}{32}$.

   (b) Let A denote the event that the least significant bit (LSB) is a ZERO. What is $P(A|B)$?
   Solution: $P(A|B) = \frac{P(AB)}{P(B)}$. To calculate $P(AB)$ note that event AB is the set of outcomes in which the memory word has exactly 3 ZEROes with one ZERO in the LSB. Thus, the 7 non-LSB bits have exactly 5 ONEs and 2 ZEROs. Therefore, $|AB| = \frac{7!}{5! \times 2!} = 21$ and therefore $P(AB) = \frac{21}{256}$ and $P(A|B) = \frac{3}{8}$.

2. **[10+6+8 points]** A 3-faced die is rolled twice. $\omega_1 \in \{1, 2, 3\}$ and $\omega_2 \in \{1, 2, 3\}$ are the outcomes of each roll. The random variable X takes values in the set $\{2\omega_1 + \omega_2\}$.

   (a) Determine the set of values u_i that X can take and the probability mass function $P_X(u_i)$.
   Solution: The sample space

   $$\Omega = \{(\omega_1, \omega_2\} = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

   and $\{2\omega_1 + \omega_2\} = \{3, 4, 5, 5, 6, 7, 7, 8, 9\}$ Therefore, $P_X(3) = \frac{1}{9}$, $P_X(4) = \frac{1}{9}$, $P_X(5) = \frac{2}{9}$, $P_X(6) = \frac{1}{9}$, $P_X(7) = \frac{2}{9}$, $P_X(8) = \frac{1}{9}$, $P_X(9) = \frac{1}{9}$.

   (b) Calculate $E[X]$.
   Solution: $E[X] = (3 + 4 + 6 + 8 + 9) \times \frac{1}{9} + (5 + 7) \times \frac{2}{9} = \frac{18}{3}$.

   (c) Calculate $\text{Var}(X)$.
   Solution: $E[X^2] = (3^2 + 4^2 + 6^2 + 8^2 + 9^2) \times \frac{1}{9} + (5^2 + 7^2) \times \frac{2}{9} = \frac{354}{9}$. Therefore $\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{354-324}{9} = \frac{10}{3}$.

3. **[8+10+4 points]** Suppose 50% of total circulating quarter coins in the US are State Quarters (see the figure below for three examples of State Quarters). Suppose that the State Quarters of each State take 1% of the total circulating quarters. Therefore, randomly picking a circulating quarter, the probability that the quarter is an Illinois State (or any other state) Quarter is 1%, and the probability that the quarter is a State Quarter is 50%. (These percentages are made up for exam purposes only.) Bob is collecting the State Quarters by checking every quarter coin he receives in his daily life.

   (a) What is the expected number of quarters that Bob needs to check to collect 50 Illinois State quarters?
   Solution: Let X denote the number of quarters Bob needs to check to collect r Illinois State quarters. X follows a negative binomial distribution with parameters $r = 50$ and $p = 0.01$.
   $E[X] = r/p = 50/0.01 = 5,000$.

Figure 1: Examples of state quarter coins

(b) What is the expected number of quarters Bob needs to check to collect two quarters of different states? Present your answer in an irreducible fraction.

Solution: Let Y denote the number of quarters Bob needs to check to collect two quarters of different states. Let Y_1 denote the number of quarters Bob needs to check to collect one State quarter. Let Y_2 denote the additional number of quarters Bob needs to check to collect another quarter of a different state. Y_1 follows a geometric distribution with parameter $p = 0.5$, and Y_2 follows a geometric distribution with parameter $p = 0.49$. $Y = Y_1 + Y_2$
$E[Y] = E[Y_1] + E[Y_2] = 1/0.5 + 1/0.49 = 198/49$.

(c) What is the expected number of quarters that Bob needs to check to collect a set of state quarters of all 50 states? Present your answer with an integer. (Hint: $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \approx \ln(n)$ and $\ln(50) \approx 3.91$)

Solution: Let Z denote the number of quarters Bob needs to check to collect a set of state quarters of all 50 states, and $Z = Z_1 + Z_2 + ... + Z_{50}$, where Z_1 denotes the number of quarters Bob needs to check to collect one state quarter, Z_i denotes the additional number of quarters Bob needs to check to collect a state quarter of one of the remaining $50 - i + 1$ states for $2 \le i \le 50$. Z_i follows a geometric distribution with parameter $p_i = \frac{50-i+1}{100}$.

$$\begin{aligned} E[Z] &= E[Z_1] + E[Z_2] + \cdots + E[Z_3] \\ &= \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_{50}} \\ &= \frac{100}{50} + \frac{100}{49} + \cdots + \frac{100}{1} \\ &= 100 \times (\frac{1}{50} + \cdots + \frac{1}{2} + \frac{1}{1}) \\ &\approx 100 \times \ln(50) \approx 391 \end{aligned}$$

4. **[8+8 points]** Suppose a random variable X is uniformly distributed on $\{2, 4, 6, 8, 10, ..., 2n\}$, and another random variable Y has a Poisson distribution with parameter λ.

(a) We observe $X = 10$. Find the ML estimate of n.

Solution: Since X has a uniform distribution, $P(X = 10) = \frac{1}{n}$, where $2n \ge 10$. To maximize $P(X = 10)$, we have $n = 5$.

(b) We observe $Y = 5$. Find the ML estimate of λ.

Solution: $P(Y = 5) = \frac{e^{-\lambda}\lambda^5}{5!}$. Differentiate with respect to λ, we obtain $\lambda = 5$.

5. **[12+6 points]** Suppose Z_1, Z_2, Z_3 are i.i.d. Bernoulli random variables with parameter $p = \frac{1}{2}$. The random variable $S = Z_2 Z_3$ if $Z_1 = 1$, and $S = Z_2 + Z_3$ if $Z_1 = 0$.

(a) Find $P(S = 1)$.

Solution:

$$\begin{aligned} P(S=1) &= P(Z_1=1, Z_2=1, Z_3=1) + P(Z_1=0, Z_2=1, Z_3=0) \\ &\quad + P(Z_1=0, Z_2=0, Z_3=1) = \frac{3}{8}. \end{aligned}$$

(b) Find $P(Z_1 = 1 | S = 1)$.
Solution:

$$P(Z_1 = 1 | S = 1) = \frac{P(Z_1=1, Z_2=1, Z_3=1)}{P(S=1)} = \frac{\frac{1}{8}}{\frac{3}{8}} = \frac{1}{3}.$$