**University of Illinois** Spring 2018

# ECE 313: Conflict Final Exam

Thursday, May 10, 2018
1:30 p.m. — 4:30 p.m.

1. **[22 points]** Consider a continuous-type random variable, $X$, with pdf of the form:

$$f_X(x) = \begin{cases} A, & x < 0, \\ x^2, & 0 \leq x < 1, \\ B(2-x)^2, & 1 \leq x < 2, \\ C, & x \geq 2, \end{cases} \tag{1}$$

where $A$, $B$, and $C$ are constants.

(a) Compute the values of $A$, $B$, and $C$, that make $f_X$ a valid pdf, and plot $f_X(x)$.
**Solution:** For $f_X(x)$ to be a valid pdf, it must satisfy:

$$\begin{aligned} 1 &= \int_{-\infty}^{0} A dx + \int_0^1 x^2 dx + \int_1^2 B(2-x)^2 dx + \int_0^{\infty} C dx \\ &= -A\infty + \frac{1}{3} + B\frac{1}{3} + C\infty. \end{aligned} \tag{2}$$

Clearly, $A = C = 0$, from where it follows that $1 = \frac{1}{3} + B\frac{1}{3}$, which yields $B = 2$.

(b) Derive the CDF, $F_X$, corresponding to $f_X$.
**Solution:**

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{x^3}{3}, & 0 \leq x < 1, \\ 1 - 2\frac{(2-x)^3}{3}, & 1 \leq x < 2, \\ 1, & x \geq 2, \end{cases} \tag{3}$$

(c) Compute $P\{X \leq 1\}$.
**Solution:**

$$\begin{aligned} P\{X \leq 1\} &= F_X(1) \\ &= \frac{1}{3}. \end{aligned} \tag{4}$$

2. **[16 points]** Consider random variables $X \sim N(\mu, \sigma^2)$, and $Y = 2X + 5$.

(a) Find $\mu$ and $\sigma^2$ if $Var(Y) = 8$ and $P(Y < 6) = 1/2$.
**Solution:** $Var(Y) = 4Var(X) = 8$, and therefore $Var(X) = \sigma^2 = 2$.
Since $P(Y < 6) = 1/2$, $E[Y] = 6 = 2E[X] + 5$, and therefore $E[X] = \mu = 0.5$.

(b) For this part, assume $\mu = -0.5$ and $\sigma^2 = 1$. Compute $P(Y^2 - 2Y \leq 0)$, and leave your answer as a function of the $Q$ function.

**Solution:** We have $E[Y] = 2E[X] + 5 = 4$ and $Var(Y) = 4Var(X) = 4$.

$$\begin{aligned} P(Y^2 - 2Y \leq 0) &= P(0 \leq Y \leq 2) \\ &= P(\frac{0-4}{2} \leq \frac{Y-4}{2} \leq \frac{2-4}{2}) \\ &= P(-2 \leq \frac{Y-4}{2} \leq -1) \\ &= Q(1) - Q(2) \end{aligned}$$

3. **[24 points]** Given $n$ MPs submitted by ECE students. An MP contains a nasty bug with probability $p$, independent of other MPs. We run the MPs one by one on our system. The system crashes if an MP contains a nasty bug.

   (a) If $n = 3$ and $p = 0.5$, find the probability that the system crashes.
   **Solution:** Let $X$ be the number of MPs that contain a nasty bug. $X$ is a binomial random variable with $p = 0.5$.

   $$P(X > 0) = 1 - P(X = 0) = 1 - (0.5)^3 = \frac{7}{8}.$$

   (b) If $p = 0.2$, find the probability that the system crashes on the 4th MP.
   **Solution:** Let $Y$ be a geometric random variable with $p = 0.2$.

   $$P(Y = 4) = 0.2(1 - 0.2)^3 = \frac{64}{625}.$$

   (c) If $p = 0.2$, given the system did not crash after running three MPs, what is the probability that it crashes on the 5th MP?
   **Solution:**

   $$P(Y = 5|Y > 3) = (1 - p)p = \frac{4}{25}.$$

   (d) If $n = 100$ and $p = 0.01$, find the Poisson approximation of the probability that the system crashes. Leave your answer in terms of powers of $e$, e.g., $ae^b$.
   **Solution:**

   $$P(X > 0) = 1 - e^{-\lambda} = 1 - e^{-1}.$$

4. **[18 points]** The two parts of the problem are unrelated.

   (a) Let $X$ be a random variable with CDF

   $$F_X(u) = \begin{cases} 0 & u \leq 1 \\ \frac{1}{2}u - \frac{1}{2} & 1 < u < 3 \\ 1 & u \geq 3 \end{cases}.$$

   Let $Y = -2X + 3$. Obtain $F_Y(v)$, the CDF of $Y$, for all $v$.
   **Solution:** First, notice that $1 < X < 3$ implies that $-3 < Y = -2X + 3 < 1$. Then, for any $v \in (-3, 1)$,

   $$F_Y(v) = P\{Y \leq v\} = P\{-2X + 3 \leq v\} = P\left\{X \geq \frac{v-3}{-2}\right\} = 1 - F_X\left(\frac{v-3}{-2}\right).$$

Hence,

$$F_Y(v) = \begin{cases} 0 & v \le -3 \\ \frac{1}{4}v + \frac{3}{4} & -3 < v < 1 \\ 1 & v \ge 1 \end{cases}.$$

(b) Let $X$ be a random variable with pdf $f_X(u) = \frac{2u}{a^2} + \frac{2}{a}$ for $u \in (-a, 0)$ and zero otherwise, for some real number $a$. The experiment is performed, and it is observed that $X = -\frac{1}{3}$. Determine $\hat{a}_{ML}$, the maximum likelihood value of the parameter $a$.
**Solution:** The objective is to maximize $f_X\left(-\frac{1}{3}\right)$, the likelihood of observing $X = -\frac{1}{3}$. Taking derivatives,

$$0 = \frac{d}{da} f_X\left(-\frac{1}{3}\right) = \frac{d}{da}\left(\frac{2}{a^2}\left(-\frac{1}{3}\right) + \frac{2}{a}\right) = \frac{4}{3a^3} - \frac{2}{a^2}.$$

Solving for $a$ yields $a \in \{0, \frac{2}{3}\}$. Hence, for us to be able to observe $X = -\frac{1}{3}$, we need $\hat{a}_{ML} = \frac{2}{3}$.

5. **[12 points]** Let $X$ be a geometric random variable with $p = 0.2$.

(a) Find $E[X]$.
**Solution:** $E[X] = \frac{1}{0.2} = 5$.

(b) Find $E[X|X > 2]$.
**Solution:** $E[X|X > 2] = \frac{1}{0.2} + 2 = 7$.

6. **[18 points]** Consider the following communication channel, with $X$ and $Z$ zero mean and uncorrelated, and $Y = b(aX + Z)$, with $a$ and $b$ constants.

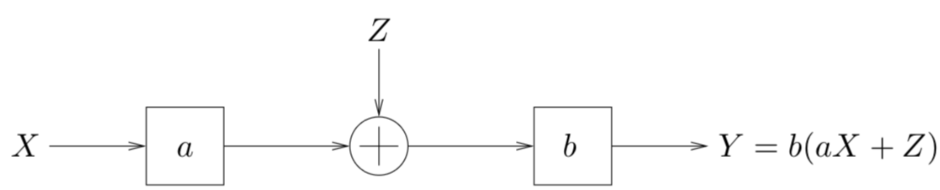


Figure 1:

(a) Find the MMSE linear estimate of X given Y in terms only of $\sigma_X$, $\sigma_Z$, $a$ and $b$.
**Solution:** The linear estimator is given by

$$\hat{X} = \frac{Cov(X, Y)}{Var(Y)}(Y - E[Y]) + E[X].$$

We have

$$E(Y) = E[b(aX + Z)] = abE[X] + bE[Z] = 0$$

$$\begin{aligned} Var(Y) &= E[Y^2] - E[Y]^2 \\ &= E[(abX + bZ)^2] \\ &= a^2b^2E[X^2] + 2ab^2E[XZ] + b^2E[Z^2] \\ &= a^2b^2\sigma_X^2 + b^2\sigma_Z^2 \end{aligned}$$

$$\begin{aligned} Cov(X,Y) &= Cov(X, abX + bZ) \\ &= ab\sigma_X^2 \end{aligned}$$

Therefore, we have

$$\hat{X} = \frac{ab\sigma_X^2}{a^2b^2\sigma_X^2 + b^2\sigma_Z^2} Y$$

(b) Find the corresponding MSE for the linear estimate computed above.
**Solution:**
The MMSE is given by

$$Var(X) - \frac{(Cov(X,Y))^2}{Var(Y)} = \sigma_X^2 - \frac{a^2b^2\sigma_X^4}{a^2b^2\sigma_X^2 + b^2\sigma_Z^2}$$

7. [ **15 points**] The two parts of the problem are unrelated.

(a) Suppose a fair die is rolled 100 times. What is a rough approximation to the sum of the numbers showing, based on the law of large numbers?
**Solution:** This is problem 4.10.2 from the text, with the value 1000 changed to 100. Since we have $E[X_i] = \frac{1+2+3+4+5+6+}{6} = 3.5$ for each role outcome variable $X_i$, $i = 1, \ldots, 100$, the law of large numbers asserts that we expect to see a value close to $100 \cdot 3.5 = 350$.

(b) Suppose each of 1200 real numbers are rounded to the nearest integer and then added. Assume the individual roundoff errors are independent and uniformly distributed over the interval $[-0.5, 0.5]$. The random variable equal to the sum is denoted by $S$. Using the CLT, find the approximate probability that the absolute value of the sum of the errors is greater than 5.
**Solution:** This is problem 4.10.7 from your text, with small numerical changes. The uniform random variables $X_i, i = 1, \ldots, 120$ to be summed up have expected value $X_i = 0$ for all $i$, and $\text{var}(X_i) = \frac{1}{12}$. Hence, the sum of the variables $S$ has expected value $E[S] = 0$ and variance $\text{var}(S) = 120 \cdot \frac{1}{12} = 100$, since the variables are independent. Hence, by the CLT,

$$P\{|S| \geq 5\} = P\left\{\frac{|S|}{10} \geq \frac{5}{10}\right\} = 2Q(0.5).$$

8. [**15 points**] The two parts of the problem are unrelated.

(a) An urn contains 990 blue balls and 10 red balls. Six students each pick a ball independently at random, with replacement, and observe its color. We wish to bound the probability that at least one student picked and observed a red ball. Let $S_k$ denote the event that student $k$ draws a red ball. Note that the probability of interest can be written as $P\{\cup_{k=1}^6 S_k\}$. Use the union bound when evaluating the probability. Compute the exact value of $P\{\cup_{k=1}^6 S_k\}$ and compare the results. You may use the fact that $0.99^6 = 0.942$ to aid your comparison.
**Solution:** By the union bound, $P\{\cup_{k=1}^6 S_k\} \leq \sum_{k=1}^6 P\{S_k\}$. Since the balls are drawn with replacement, and in each draw we have probability $10/1000 = 0.01$ to draw a red ball, the desired bound equals 0.06. The correct result may be inferred by noting that the probability we seek equals $1-p$, where $p$ is the probability that no red ball is picked. Consequently, $p = 0.99^6 = 0.942$, and $1 - p = 0.058$.

(b) An urn contains eight blue balls and four green balls. Three balls are drawn from this urn without replacement. Compute the probability that all three balls are blue.

**Solution:** The probability of drawing a blue ball the first time is equal to 8/12. The probability of drawing a blue ball the second time given that the first ball is blue is 7/11. Finally, the probability of drawing a blue ball the third time given that the first two balls are blue is 6/10. Hence, the probability of drawing three blue balls equals the product $8/12 \cdot 7/11 \cdot 6/10 = 14/55$.

9. **[15 points]** Suppose $X$ and $Y$ are zero-mean unit-variance jointly Gaussian random variables.

(a) If $\rho_{X,Y} = 0.2$, find the numerical value of $E[Y|X = 5]$.

**Solution:** Since $X$ and $Y$ are jointly Gaussian,

$$E[Y|X = 5] = \hat{E}[Y|X = 5] = \rho_{X,Y}X = 1.$$

(b) If $\rho_{X,Y} = 0$, find $f_{Y|X}(v|u = 0)$ for all $v$.

**Solution:** Since $X$ and $Y$ are jointly Gaussian, $\rho_{X,Y} = 0$ implies that they are independent. Hence,

$$f_{Y|X}(v|u = 0) = f_Y(v) = \frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}.$$

10. **[15 points]** You are given two hypothesis, $H_0$ and $H_1$, and the corresponding conditional distributions of the observed random variable $X$ given the hypotheses, as shown in the table. Here, $c \in (-\frac{1}{2}, \frac{1}{2})$ is some constant.

Table 1: The conditional distributions of $X$ given $\mathrm{H}_0$ and $\mathrm{H}_1$, respectively.

| X | 0 | 1 | 2 |
|---|---|---|---|
| $\mathrm{H}_0$ | $\frac{1}{2}+c$ | $\frac{1}{2}-c$ | 0 |
| $\mathrm{H}_1$ | 0 | $\frac{1}{2}+c$ | $\frac{1}{2}-c$ |

(a) Describe the ML decision rule. Your answer will depend on the value of $c$.

**Solution:** Clearly, in the ML rule, when $X = 0$ is observed we should decide in favor of $H_0$ and when $X = 2$ we should decide in favor of $\mathrm{H}_1$. The only question remains which decision to make when $X = 1$. Clearly,

$$\frac{P\{X = 1|H_1\}}{P\{X = 1|H_0\}} \geq 1$$

holds if and only if $c \in [0, \frac{1}{2})$, in which case we decide in favor of hypothesis $\mathrm{H}_1$. Otherwise, when $c \in (-\frac{1}{2}, 0)$ we decide in favor of hypothesis $\mathrm{H}_0$. Note that we could have broken the tie arbitrarily for $c = 0$ - in this case, we chose to break the tie in favor of hypothesis $\mathrm{H}_1$.

(b) Let $c = 0$. Find the probability of miss, false alarm and average error probability of the ML estimator (when computing the average error probability, assume that the hypothesis are equally likely.)

**Solution:** Note that if $c = 0$ we may decide in favor of $\mathrm{H}_1$ or $\mathrm{H}_0$ while breaking the ties. We opt for the former. Clearly, we only make an error if $X = 1$ is observed, in which case we have $P_{fa} = P\{\text{Decide } H_1|H_0\} = \frac{1}{2}$. Similarly, we have $P_{miss} = P\{\text{Decide } H_0|H_1\} = 0$, which gives $P_{error} = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$.

11. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Let $X$ denote a continuous-type random variable with PDF $f_X(x)$ and CDF $F_X(x)$.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $f_X(x) \leq F_X(x) \leq 1$ for all $x$. |
| ☐ | ☐ | $P\{X = 5\} = F_X(5)$. |

**Solution:** False, False

(b) Let $D$, $E_1$, and $E_2$ denote three arbitrary events of some probability space. Assume that $E_1$ and $E_2$ are mutually exclusive.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $P(E_1 E_2) = P(E_1)P(E_2)$. |
| ☐ | ☐ | $P(D) = P(D \mid E_1)P(E_1) + P(D \mid E_2)P(E_2)$. |

**Solution:** False,False

(c) Consider a binary hypothesis testing problem with some known prior distribution $(\pi_0, \pi_1)$. Let $p_{e,MAP}$ and $p_{e,ML}$ be the average probability of error of the MAP and ML rules, respectively.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $p_{e,MAP} \leq p_{e,ML}$. |
| ☐ | ☐ | $p_{false-alarm} + p_{miss} \leq 1$ for the ML rule. |
| ☐ | ☐ | $p_{miss} + p_{true-positive} = 1$ where $p_{true-positive} = P($ declare $H_1$ true$\mid H_1)$. |

**Solution:** True,True,True

(d) Consider any three events, $A$, $B$ and $C$, on a common probability space.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $P(A \bigcup B \mid C) \geq P(A \bigcup B)$. |
| ☐ | ☐ | If $P(AB \mid C) = P(AB)$, then $A$, $B$ and $C$ are independent. |
| ☐ | ☐ | If $A$, $B$ and $C$ are independent, then $P(A \mid C) = P(A \mid B)$. |

**Solution:** False,False,True