University of Illinois Spring 2018

ECE 313: Hour Exam II

Wednesday, April 11, 2018
8:45 p.m. — 10:00 p.m.

1. [**22 points**] Emails arrive to Professor X's inbox according to a Poisson process with rate $\lambda = 0.2$ emails per minute. Measure time in minutes and consider a time interval beginning at time $t = 0$. Find the probability of the following events. (Leave your answers in powers of e, e.g. ae^b for some constants a and b.)

   (a) E_1: Three emails arrive within the first minute.
   Solution: Since $\lambda = 0.2$ emails per minute, $N_1 \sim$ **Poisson**(0.2), hence

$$\begin{aligned} P(E_1) &= P\{N_1 = 3\} \\ &= \frac{(0.2)^3 e^{-0.2}}{3!} \\ &= \frac{e^{-0.2}}{750} \end{aligned} \tag{1}$$

   (b) E_3: The time it takes for the first email to arrive is at least 5 minutes.
   Solution: Since $\lambda = 0.2$ emails per minute, we have that $T_1 \sim$ **Exponenial**(0.2), hence

$$\begin{aligned} P(E_3) &= P\{T_1 \geq 5\} \\ &= e^{-0.2\times 5} \\ &= e^{-1}. \end{aligned} \tag{2}$$

   (c) E_3: The time it takes for the second email to arrive is at least 2 minutes.
   Solution: Since $\lambda = 0.2$ emails per minute, we have that $Y_2 \sim$ **Erlang**(0.2), hence

$$\begin{aligned} P(E_3) &= P\{Y_2 \geq 2\} \\ &= P\{N_2 \leq 1\} \\ &= P\{N_2 = 0\} + P\{N_2 = 1\} \\ &= e^{-0.4} + \frac{0.4e^{-0.4}}{1!} \\ &= 1.4e^{-0.4} \end{aligned} \tag{3}$$

2. [**18 points**] The random variable X has the $N(-6, 36)$ distribution. Express the answers to the following questions in terms of the Q function.

   (a) Obtain $P\{|X| > 6\}$.
   Solution: $|X| > 6$ when $X > 6$ or $X < -6$

$$\begin{aligned} P\{|X| > 6\} &= P\{X > 6\} + P\{X < -6\} \\ &= P\left\{\frac{X-(-6)}{\sqrt{36}} > \frac{6-(-6)}{\sqrt{36}}\right\} + P\left\{\frac{X-(-6)}{\sqrt{36}} < \frac{-6-(-6)}{\sqrt{36}}\right\} \\ &= Q\left(2\right) + \Phi\left(0\right) = Q\left(2\right) + \frac{1}{2}. \end{aligned}$$

(b) Let $Y = aX + b$, where $P\{2 < Y\} = \frac{1}{2}$. Determine a valid pair of values of $a \neq 0$ and b.
Solution: From the linear scaling of X, we know that Y must be Gaussian too, and hence $P\{2 < Y\} = \frac{1}{2}$ means that $2 = E[Y] = E[aX + b] = a\mu_X + b = -6a + b$ Hence, any pair (a, b) such that $6a + 2 = b$ is valid, e.g. $a = 1$ and $b = 8$.

3. **[22 points]** Suppose X and Y are independent random variables such that X is uniformly distributed over the interval $[0, 1]$ and Y is exponentially distributed with parameter $\lambda > 0$.

(a) Find the joint CDF $F_{X,Y}$ for all (u, v).
Solution: Since X and Y are independent, within the support,

$$F_{X,Y}(u, v) = P(X \leq u, Y \leq v) = P(X \leq u)P(Y \leq v) = u(1 - e^{\lambda v}).$$

Together,

$$F_{X,Y}(u, v) = \begin{cases} 0, & \text{if } u < 0 \text{ or } v < 0 \\ u(1 - e^{\lambda v}), & \text{if } u \in [0, 1],\ v \geq 0 \\ 1 - e^{\lambda v}, & \text{if } u > 1,\ v > 0 \end{cases}$$

(b) Find $P(Y = X)$.
Solution: Since X and Y are continuous random variables, $P(Y = X) = 0$.

(c) Find $P(Y \leq 4X)$.
Solution:

$$\begin{aligned} P(Y \leq 4X) &= \int_0^1 \int_0^{4u} \lambda e^{-\lambda v} dv du \\ &= \int_0^1 [-e^{-\lambda v}]_0^{4u} du \\ &= \int_0^1 1 - e^{-4\lambda u} du \\ &= \left[u + \frac{1}{4\lambda} e^{-4\lambda u} \right]_0^1 \\ &= 1 + \frac{1}{4\lambda} e^{-4\lambda} - \frac{1}{4\lambda}. \end{aligned}$$

4. **[18 points]** The two parts of the problem are unrelated.

(a) The lifetimes of light bulbs (in years) produced by two companies, *Fos* and *Illuminati*, follow exponential distributions with parameters $\lambda = 1$ and $\lambda = 2$, respectively. You purchased a random lightbulb from a store that does not carry manufacturer labels, but carries the same number of products by *Fos* and *Illuminati*. What is the probability that your lightbulb will work one year after purchase, given all the available lightbulbs in the store have been there for a year already? (Leave your answers in powers of e, e.g. ae^b for some constants a and b.)
Solution: Let X be the lifetime of the lightbulb we buy. We seek

$$
\begin{aligned}
P(X>2|X>1) &= \frac{P(X>2, X>1)}{P(X>1)} \\
&= \frac{P(X>2)}{P(X>1)} \\
&= \frac{P(X>2|\text{Fos})P(\text{Fos}) + P(X>2|\text{Ill})P(\text{Ill})}{P(X>1|\text{Fos})P(\text{Fos}) + P(X>1|\text{Ill})P(\text{Ill})} \\
&= \frac{0.5e^{-2} + 0.5e^{-4}}{0.5e^{-1} + 0.5e^{-2}} \\
&= \frac{e^{-2} + e^{-4}}{e^{-1} + e^{-2}}
\end{aligned}
$$

A common mistake was to apply the memoryless property to the first term, yielding

$$\frac{1}{2}P\{X_1 > 1\} + \frac{1}{2}P\{X_2 > 1\} = \frac{(\exp(-1) + \exp(-2))}{2}.$$

(b) Let random variable X be uniform in the interval $[0, 3]$. Show how to generate random variable Y with pmf as defined below based on X.

$$p_Y(k) = \begin{cases} 0.5, & k = 0 \\ 0.4, & k = 1 \\ 0.1, & k = 2 \end{cases}$$

Solution: Since X is uniform, we just need to divide the interval $[0, 3]$ into 3 regions such that the probability of each region matches those of Y. One possible way to do it is:

$$Y = \begin{cases} 0, & \text{if } X \in [0, 1.5] \\ 1, & \text{if } X \in [1.5, 2.7] \\ 2, & \text{if } X \in [2.7, 3] \end{cases}$$

Note that $P(X \in [0, 1.5]) = 0.5$, $P(X \in [1.5, 2.7]) = 0.4$, and $P(X \in [2.7, 3]) = 0.1$.

5. **[20 points]** The two parts of the problem are unrelated.

Hint: The derivative of the arcsin(x) function equals $\frac{1}{\sqrt{1-x^2}}$.

(a) A real-valued continuous random variable X is said to have an arcsin(-1,1) distribution if it has a CDF of the form

$$F_X(x) = c \arcsin\left(\sqrt{\frac{x+1}{2}}\right),$$

where $x \in (-1, 1)$, and c is some real-valued constant. Recall that arcsin(a) stands for the inverse sin function, that is, a function that returns the angle (in radians) whose sin equals a.

Determine the constant c and describe the values of the CDF $F_X(x)$ outside of the interval $(-1, 1)$ that will make it into a valid CDF. Find the pdf $f_X(x)$ of X and determine the values of the pdf for $x = -1$ and $x = 1$. Why are these values allowed?

Solution: Clearly, the CDF cannot be complex-valued and since $\lim_{x\to\infty} F_X(x) = 0$, we have $F_X(x) = 0$ for all $x \leq -1$. Also, since $\arcsin(1) = \frac{\pi}{2}$, we have $c = \frac{2}{\pi}$. Consequently, $F_X(x) = 1$ for all $x \geq 1$. Using the product rule for the derivative, it is straightforward to see that

$$f_X(x) = \frac{1}{\pi\sqrt{(1-x)(x+1)}},$$

for $x \in (-1, 1)$ and zero elsewhere.

Note that the values of the pdf at $x = -1$ and $x = 1$ are unbounded, but that does not invalidate any of the pdf properties as any individual point on the real line has probability zero.

(b) Let U be a random variable uniformly distributed in $[-\pi, \pi]$. Find the CDF and pdf of the random variable $X = \sin(U)$.

Solution: Two related problems have been solved in your textbook, starting on page 129. The only difference is that the support of the random variable U is $[0, \pi]$ and that the function is $\cos(x)$ and not $\sin(x)$. Your solution should be the arcsin distribution from the first part of the problem.