University of Illinois Spring 2018

ECE 313: Hour Exam II

Wednesday, April 11, 2018
8:45 p.m. — 10:00 p.m.

1. **[22 points]** A shuttle bus arrives at a bus stop at 12:00PM and leaves at 12:02PM. People arrive at the bus stop to catch the shuttle according to a Poisson process with rate $\lambda = 0.1$ person per minute. Assuming there are no people waiting for the shuttle bus upon its arrival at 12:00PM, find the probability of the following events. (Leave your answers in powers of e, e.g. ae^b for some constants a and b.)

   (a) E_1: The shuttle does not pick any new passenger at the bus stop.
   Solution: Since the time the shuttle spends at the bus stop is 2 minutes, and the number arriving to the bus stop is Poisson with rate $\lambda = 0.1$, the number of people arriving in the two minutes that shuttle is $N_2 \sim \mathbf{Poisson}(0.2)$. Thus,

$$\begin{aligned} P(E_1) &= P\{N_2 = 0\} \\ &= e^{-0.2}. \end{aligned} \tag{1}$$

   (b) E_2: The shuttle picks up two new passengers at the bus stop.
   Solution:

$$\begin{aligned} P(E_2) &= P\{N_2 = 2\} \\ &= \frac{(0.2)^2 e^{-0.2}}{2!} \\ &= \frac{e^{-0.2}}{50} \end{aligned} \tag{2}$$

   (c) E_3: The only passenger the shuttle picks arrives at the bus stop after 12:01PM.
   Solution:

$$\begin{aligned} P(E_3) &= P(N_1 = 0)P(N_1 = 1) \\ &= 0.2e^{-0.2}e^{-0.2} \\ &= 0.2e^{-0.4}. \end{aligned} \tag{3}$$

2. **[22 points]** Suppose X and Y are independent random variables such that X is uniformly distributed over the interval $[0, 1]$ and Y is exponentially distributed with parameter $\lambda > 0$.

   (a) Find the joint CDF $F_{X,Y}$.
   Solution: Since X and Y are independent, within the support,

$$F_{X,Y}(u, v) = P(X \le u, Y \le v) = P(X \le u)P(Y \le v) = u(1 - e^{\lambda v}).$$

   Together,

$$F_{X,Y}(u, v) = \begin{cases} 0, & \text{if } u < 0 \text{ or } v < 0 \\ u(1 - e^{\lambda v}), & \text{if } u \in [0, 1],\ v \ge 0 \\ 1 - e^{\lambda v}, & \text{if } u > 1,\ v > 0 \end{cases}$$

(b) Find $P(Y = X)$.
Solution: Since X and Y are continuous random variables, $P(Y = X) = 0$.

(c) Find $P(Y \leq 4X)$.
Solution:

$$\begin{aligned} P(Y \leq 4X) &= \int_0^1 \int_0^{4u} \lambda e^{-\lambda v} dv du \\ &= \int_0^1 [-e^{-\lambda v}]_0^{4u} du \\ &= \int_0^1 1 - e^{-4\lambda u} du \\ &= \left[u + \frac{1}{4\lambda} e^{-4\lambda u} \right]_0^1 \\ &= 1 + \frac{1}{4\lambda} e^{-4\lambda} - \frac{1}{4\lambda}. \end{aligned}$$

3. **[18 points]** Consider a binary hypothesis testing problem where

$$H_0: \quad f_0(y) = \begin{cases} y+1, & y \in (-1,0) \\ -y+1, & y \in (0,1) \\ 0, & \text{else.} \end{cases}$$
$$H_1: \quad f_1(y) = \begin{cases} \frac{1}{3}, & y \in (0,3) \\ 0, & \text{else.} \end{cases},$$

(a) Determine the ML decision rule.
Solution: For the ML decision rule, we decide H_1 when

$$\Lambda(y) = \frac{f_1(y)}{f_0(y)} \geq 1,$$

which in this case yields, for $y \in (0,1)$,

$$\frac{\frac{1}{3}}{-y+1} \geq 1.$$

This simplifies to $y \geq \frac{2}{3}$. However, under H_1, $y \leq 3$, hence, the ML rule decides H_1 when $y \in \left[\frac{2}{3}, 3\right]$ and H_0 else.

(b) Forget about the result in part (a). Determine a decision rule that yields $p_{miss} = \frac{5}{6}$.
Solution: Recall that $p_{miss} = P\{\text{declare } H_0 | H_1\}$. One possible rule would be to declare H_1 if $y \in [a, 3]$, for some value of $a \in [0, 3]$.
This means that $\frac{5}{6} = p_{miss} = \int_{-1}^{a} f_1(y) dy = \frac{1}{3}(a)$, which yields $a = \frac{5}{2}$.
Note that there are other possibilities for the rule, as long as the region to declare H_1 over $y \in (0, 3)$ has a combined length of $\frac{1}{2}$.

4. **[18 points]** The two parts of the problem are unrelated.

(a) The lifetimes of light bulbs (in years) produced by two companies, *Fos* and *Illuminati*, follow exponential distributions with parameters $\lambda = 1$ and $\lambda = 2$, respectively. You purchased a random lightbulb from a store that does not carry manufacturer labels, but carries the same number of products by *Fos* and *Illuminati*. What is the probability that your lightbulb will work one year after purchase, given all the available lightbulbs in the store have been there for a year already? (Leave your answers in powers of e, e.g. ae^b for some constants a and b.)

Solution: Let X be the lifetime of the lightbulb we buy. We seek

$$\begin{aligned} P(X > 2|X > 1) &= \frac{P(X > 2, X > 1)}{P(X > 1)} \\ &= \frac{P(X > 2)}{P(X > 1)} \\ &= \frac{P(X > 2|\text{Fos})P(\text{Fos}) + P(X > 2|\text{Ill})P(\text{Ill})}{P(X > 1|\text{Fos})P(\text{Fos}) + P(X > 1|\text{Ill})P(\text{Ill})} \\ &= \frac{0.5e^{-2} + 0.5e^{-4}}{0.5e^{-1} + 0.5e^{-2}} \\ &= \frac{e^{-2} + e^{-4}}{e^{-1} + e^{-2}} \end{aligned}$$

A common mistake was to apply the memoryless property to the first term, yielding

$$\frac{1}{2}P\{X_1 > 1\} + \frac{1}{2}P\{X_2 > 1\} = \frac{(\exp(-1) + \exp(-2))}{2}.$$

(b) Let random variable X be uniform in the interval $[0, 3]$. Show how to generate random variable Y with pmf as defined below based on X.

$$p_Y(k) = \begin{cases} 0.5, & k = 0 \\ 0.4, & k = 1 \\ 0.1, & k = 2 \end{cases}$$

Solution: Since X is uniform, we just need to divide the interval $[0, 3]$ into 3 regions such that the probability of each region matches those of Y. One possible way to do it is:

$$Y = \begin{cases} 0, & \text{if } X \in [0, 1.5] \\ 1, & \text{if } X \in [1.5, 2.7] \\ 2, & \text{if } X \in [2.7, 3] \end{cases}$$

Note that $P(X \in [0, 1.5]) = 0.5$, $P(X \in [1.5, 2.7]) = 0.4$, and $P(X \in [2.7, 3]) = 0.1$.

5. **[20 points]** The two parts of the problem are unrelated.

Hint: The derivative of the arcsin(x) function equals $\frac{1}{\sqrt{1-x^2}}$.

(a) A real-valued continuous random variable X is said to have an arcsin(-1,1) distribution if it has a CDF of the form

$$F_X(x) = c \arcsin\left(\sqrt{\frac{x+1}{2}}\right),$$

where $x \in (-1, 1)$, and c is some real-valued constant. Recall that $\arcsin(a)$ stands for the inverse sin function, that is, a function that returns the angle (in radians) whose sin equals a.

Determine the constant c and describe the values of the CDF $F_X(x)$ outside of the interval $(-1, 1)$ that will make it into a valid CDF. Find the pdf $f_X(x)$ of X and determine the values of the pdf for $x = -1$ and $x = 1$. Why are these values allowed?

Solution: Clearly, the CDF cannot be complex-valued and since $\lim_{x \to \infty} F_X(x) = 0$, we have $F_X(x) = 0$ for all $x \leq -1$. Also, since $\arcsin(1) = \frac{\pi}{2}$, we have $c = \frac{2}{\pi}$. Consequently, $F_X(x) = 1$ for all $x \geq 1$. Using the product rule for the derivative, it is straightforward to see that

$$f_X(x) = \frac{1}{\pi \sqrt{(1-x)(x+1)}},$$

for $x \in (-1, 1)$ and zero elsewhere.

Note that the values of the pdf at $x = -1$ and $x = 1$ are unbounded, but that does not invalidate any of the pdf properties as any individual point on the real line has probability zero.

(b) Let U be a random variable uniformly distributed in $[-\pi, \pi]$. Find the CDF and pdf of the random variable $X = \sin(U)$.

Solution: Two related problems have been solved in your textbook, starting on page 129. The only difference is that the support of the random variable U is $[0, \pi]$ and that the function is $\cos(x)$ and not $\sin(x)$. Your solution should be the arcsin distribution from the first part of the problem.