University of Illinois Spring 2018

ECE 313: Hour Exam I

Wednesday, February 28, 2018
8:45 p.m. — 10:00 p.m.

1. **[12 points]** A blood test gives readings of an indicator X according to the following likelihood matrix

	$X=0$	$X=1$	$X=2$	$X=3$
H_1	0.0	0.35	0.6	0.05
H_0	0.4	0.3	0.2	0.1

The priors are $(\pi_0, \pi_1) = (0.2, 0.8)$.

(a) What is the maximum likelihood (ML) decision rule? Compute p_{miss} for the ML decision rule.
Solution: The ML decision rule:

	$X=0$	$X=1$	$X=2$	$X=3$
H_1	0.0	$\underline{0.35}$	$\underline{0.6}$	0.05
H_0	$\underline{0.4}$	0.3	0.2	$\underline{0.1}$

$$p_{\text{miss}} = 0.05.$$

(b) What is the maximum a posteriori (MAP) decision rule? Compute p_e for the MAP decision rule.
Solution: The MAP decision rule with threshold 0.25:

	$X=0$	$X=1$	$X=2$	$X=3$
H_1	0.0	$\underline{0.35}$	$\underline{0.6}$	$\underline{0.05}$
H_0	$\underline{0.4}$	0.3	0.2	0.1

$$p_{\text{miss}} = 0.$$

$$p_{\text{false alarm}} = 0.3 + 0.2 + 0.1 = 0.6.$$

$$p_e = 0.2 \times 0.6 = 0.12.$$

(c) What is the decision rule that minimizes p_{miss}?
Solution: The MAP rule minimizes p_{miss}. Or any rule that declares H_1 for $X = 1$, 2, and 3.

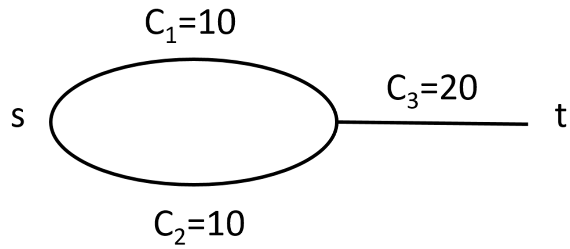
2. **[10 points]** Consider the following network. Each link fails with probability p.

(a) What is the outage probability?
Solution:

$$P(F) = p^2 + p - p^3$$

(b) What is the probability that the network has capacity 10?
Solution:

$$p_X(10) = 2p(1-p)(1-p) = 2p(1-p)^2.$$



3. **[20 points]** The three parts of this problem are unrelated.

   (a) Consider a random variable Y. It is known that $E[-3Y-2] = 4$ and that $Var(-3Y-2) = 36$. Determine $E[Y]$ and $Var(Y)$.
   Solution: By linearity of expectation, $4 = E[-3Y-2] = -3E[Y]-2,$, hence $E[Y] = -2$. By scaling of variance, $36 = Var(-3Y-2) = (-3)^2 Var(Y)$, hence $Var(Y) = 4$.

   (b) Consider rolling a fair die and flipping a fair coin. Define a random variable X which is equal to the number shown in the die if the coin shows heads and twice the number in the die if the coin shows tails. Obtain the pmf of X.
   Solution: There are 12 equally likely outcomes because each of the six outcomes from the die can occur along with tails or heads from the coin. One can then map the corresponding outcomes to the support of the pmf as $\{1, 2, 3, 4, 5, 6, 8, 10, 12\}$, and the corresponding pmf is given by

$$p_X(k) = \begin{cases} \frac{1}{6} & k = 2, 4, 6 \\ \frac{1}{12} & k = 1, 3, 5, 8, 10, 12 \\ 0 & else \end{cases}$$

   (c) Let Z be a random variable with pmf $p_Z(k) = \frac{c}{(2^k)}$ for $k \in \{1, 2, 3, 4\}$ and zero else. Determine the value of the constant c, of the mean $E[Z]$ and of $E[2^Z]$.
   Solution: The pmf must add up to one, so

$$1 = \sum_k p_Z(k) = \sum_{k=1}^{4} \frac{c}{(2^k)} = c\frac{15}{16},$$

   hence $c = \frac{16}{15}$. By definition of the mean,

$$E[Z] = \sum_k k p_Z(k) = \sum_{k=1}^{4} k \frac{c}{(2^k)} = c\frac{13}{8} = \frac{26}{15}.$$

   By LOTUS,

$$E[2^Z] = \sum_k 2^k p_Z(k) = \sum_{k=1}^{4} 2^k \frac{c}{(2^k)} = c4 = \frac{64}{15},$$

4. **[18 points]** If an intercontinental ballistic missile (ICBM) is launched from a nuclear submarine in a certain part of the world, a radar system might be able to detect its presence while airborne and launch a second missile to intercept the ICBM. Let M denote the event that an ICBM has been launched, and assume $P(M) = 0.001$. Let D denote the event that the radar system alerts of the presence of a missile, and assume $P(D|M) = 0.99$. In addition, assume that $P(D^c|M^c) = 0.99$.

(a) Let E denote the event that the radar system makes a mistake. Compute $P(E)$.
Solution:

$$\begin{aligned}P(E) &= P(E|M)P(M) + P(E|M^c)P(M^c)\\ &= P(D^c|M)P(M) + P(D|M^c)P(M^c)\\ &= 0.01 \times 0.001 + 0.01 \times 0.999\\ &= 0.01\end{aligned}$$

(b) Given that the radar system detects the presence of an ICBM, compute the probability that an ICBM has actually been launched.
Solution: We can use the total probability theorem as follows:

$$\begin{aligned}P(D) &= P(D|M)P(M) + P(D|M^c)P(M^c)\\ &= 0.99 \times 0.001 + 0.01 \times 0.999\\ &= 0.01098\end{aligned}$$

$$\begin{aligned}P(M|D) &= \frac{P(D|M)P(M)}{P(D)}\\ &= \frac{0.00099}{0.01098}\\ &= 0.0902.\end{aligned}$$

5. **[22 points]** This problem considers two basketball teams A and B. The three parts are unrelated.

(a) We estimate the probability that team A beats team B (denoted by p) by having them play against each other n times. If we want to estimate p within 0.1 and with 75% confidence (using the confidence interval based on the Chebychev bound), how many games they should play?
Solution: Recall that we have

$$P\left\{p \in \left(\hat{p} - \frac{a}{2\sqrt{n}}, \hat{p} + \frac{a}{2\sqrt{n}}\right)\right\} \geq 1 - \frac{1}{a^2}$$

For a 75% confidence, we need $a = 2$. Therefore, to estimate p within 0.1, we have that

$$\frac{2}{2\sqrt{n}} = 0.1,$$

or in other words, $n = 100$.

(b) The expected number of points made by team A on a game is 75. Provide a lower bound on the probability that team A makes less than 90 points on a given game.
Solution: Denote by X the number of points made by team A on a given game. We only know that $\mathbb{E}[X] = 75$, and hence we apply Markov Inequality to give a bound on

the probability. Specifically, we have that:

$$\begin{aligned}
P(X < 90) &= 1 - P(X \geq 90) \\
&\geq 1 - \frac{\mathbb{E}[X]}{90} \\
&= 1 - \frac{75}{90} \\
&= \frac{1}{6}
\end{aligned}$$

(c) We want to estimate the probability that team A beats team B. To that end, we make them play against each other until team A beats team B 3 times, which happens at the 10th game. What is the Maximum Likelihood (ML) estimation of the probability that team A beats team B? Assume each game is independent of previous games.

Solution: Denote by the random variable X the number of games until team A beats team B 3 times. Then X has the negative binomial distribution with parameters 3 and p, where p denotes the probability that team A beats team B.

Given that we observe $X = 10$, we have that:

$$\begin{aligned}
\hat{p}_{ML} &= \arg\max_p \left\{ \binom{10-1}{2} p^3 (1-p)^7 \right\} \\
&= \arg\max_p \{p^3(1-p)^7\} \\
&= \arg\max_p \{\ln(p^3(1-p)^7)\} \\
&= \arg\max_p \{3\ln p + 7\ln(1-p)\}
\end{aligned}$$

Taking the derivate over p and setting it to zero, we get that:

$$\frac{3}{p} - \frac{7}{1-p} = 0,$$

which is satisfied for $p = 0.3$. Therefore, $\hat{p}_{ML} = 0.3$.

6. **[18 points]** The two parts are unrelated.

(a) A basketball team is composed of 10 players, 6 of which are considered to be "very good", and 4 "average". At any given time on a basketball game, only 5 players are in the field. Let's assume that we decide on the 5 players at the beginning of the game, and that they play the whole game. Let's denote the selected 5 players as the *playing team*. We say that the *playing team* is good if at least 3 out of the 5 players are "very good". Find the probability that a *playing team* is good.

Solution: A team is good if at least 3 out of the 5 players are very good (i.e., 3, 4, or 5 players have to be very good). We have a total of 10 players to choose from, 6 of which are very good.

The total number of distinct playing teams that can be made is $\binom{10}{5}$. From those, we need to count how many are considered good, i.e., how many have at least 3 very good players.

The number of teams with 3 very good players is given by $\binom{4}{2}\binom{6}{3}$, with 4 very good players by $\binom{4}{1}\binom{6}{4}$, and with 5 very good players by $\binom{4}{0}\binom{6}{5}$.

Therefore, the probability that a playing team is good is given by

$$\frac{\binom{4}{2}\binom{6}{3} + \binom{4}{1}\binom{6}{4} + \binom{4}{0}\binom{6}{5}}{\binom{10}{5}}$$

(b) A committee of four people is to be selected from a group of four man and four women. What is the probability that the committee is not gender biased (i.e., it has the same number of men and women)?

Solution: Not being gender biased implies selecting the same number of man and women for the committee, which is two each. There is a total of $\binom{8}{4}$ choices for the committee members, out of which $\left(\binom{4}{2}\right)^2$ are favorable. Hence, the desired probability is $\frac{18}{35}$.