University of Illinois Spring 2018

ECE 313: Hour Exam I

Wednesday, February 28, 2018
8:45 p.m. — 10:00 p.m.

1. **[12 points]** A blood test gives readings of an indicator X according to the following likelihood matrix.

	$X=0$	$X=1$	$X=2$	$X=3$
H_1	0.0	0.35	0.6	0.05
H_0	0.4	0.3	0.2	0.1

The priors are $(\pi_0, \pi_1) = (0.2, 0.8)$.

(a) What is the maximum likelihood (ML) decision rule? Compute p_{miss} for the ML decision rule.
Solution: The ML decision rule:

	$X=0$	$X=1$	$X=2$	$X=3$
H_1	0.0	$\underline{0.35}$	$\underline{0.6}$	0.05
H_0	$\underline{0.4}$	0.3	0.2	$\underline{0.1}$

$$p_{\text{miss}} = 0.05.$$

(b) What is the maximum a posteriori (MAP) decision rule? Compute p_e for the MAP decision rule.
Solution: The MAP decision rule with threshold 0.25:

	$X=0$	$X=1$	$X=2$	$X=3$
H_1	0.0	$\underline{0.35}$	$\underline{0.6}$	$\underline{0.05}$
H_0	$\underline{0.4}$	0.3	0.2	0.1

$$p_{\text{miss}} = 0.$$

$$p_{\text{false alarm}} = 0.3 + 0.2 + 0.1 = 0.6.$$

$$p_e = 0.2 \times 0.6 = 0.12.$$

(c) What is the decision rule that minimizes p_{miss}?
Solution: The MAP rule minimizes p_{miss}. Or any rule that declares H_1 for $X = 1$, 2, and 3.

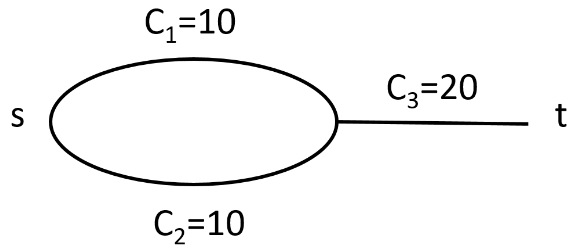
2. **[10 points]** Consider the following network. Each link fails with probability p.

(a) What is the outage probability?
Solution:

$$P(F) = p^2 + p - p^3$$

(b) What is the probability that the network has capacity 10?
Solution:

$$p_X(10) = 2p(1-p)(1-p) = 2p(1-p)^2.$$



3. **[20 points]** The three parts of this problem are unrelated.

   (a) Consider a random variable Y. It is known that $E[-3Y-2] = 4$ and that $Var(-3Y-2) = 36$. Determine $E[Y]$ and $Var(Y)$.
   Solution: By linearity of expectation, $4 = E[-3Y-2] = -3E[Y]-2,$, hence $E[Y] = -2$. By scaling of variance, $36 = Var(-3Y-2) = (-3)^2 Var(Y),$, hence $Var(Y) = 4$.

   (b) Consider rolling a fair die and flipping a fair coin. Define a random variable X which is equal to the number shown in the die if the coin shows heads and twice the number in the die if the coin shows tails. Obtain the pmf of X.
   Solution: There are 12 equally likely outcomes because each of the six outcomes from the die can occur along with tails or heads from the coin. One can then map the corresponding outcomes to the support of the pmf as $\{1, 2, 3, 4, 5, 6, 8, 10, 12\}$, and the corresponding pmf is given by

$$p_X(k) = \begin{cases} \frac{1}{6} & k = 2, 4, 6 \\ \frac{1}{12} & k = 1, 3, 5, 8, 10, 12 \\ 0 & else \end{cases}$$

   (c) Let Z be a random variable with pmf $p_Z(k) = \frac{c}{(2^k)}$ for $k \in \{1, 2, 3, 4\}$ and zero else. Determine the value of the constant c, of the mean $E[Z]$ and of $E[2^Z]$.
   Solution: The pmf must add up to one, so

$$1 = \sum_k p_Z(k) = \sum_{k=1}^{4} \frac{c}{(2^k)} = c\frac{15}{16},$$

   hence $c = \frac{16}{15}$. By definition of the mean,

$$E[Z] = \sum_k k p_Z(k) = \sum_{k=1}^{4} k \frac{c}{(2^k)} = c\frac{13}{8} = \frac{26}{15}.$$

   By LOTUS,

$$E[2^Z] = \sum_k 2^k p_Z(k) = \sum_{k=1}^{4} 2^k \frac{c}{(2^k)} = c4 = \frac{64}{15},$$

4. **[18 points]** Let X denote a random variable with a Binomial distribution with parameters $n \geq 1$ and $0 \leq p \leq 1$.

(a) Assume that $p = 3\alpha$, where $\alpha \geq 0$ is unknown. Compute the maximum likelihood estimate of α, denoted by $\hat{\alpha}_{ML}$, for the case when $n = 10$, and $X = 6$.
Solution: We want to find the value of α that maximizes the function

$$f(\alpha) = \binom{10}{6}(3\alpha)^6(1-3\alpha)^4$$

subject to $\alpha \geq 0$; we denote this value by α^*. By differentiating $f(\alpha)$, we obtain

$$\frac{df(a)}{da} = 3\binom{10}{6}(3\alpha)^5(1-3\alpha)^3(6-30\alpha),$$

and by setting this expression to zero, we obtain $\alpha = \frac{1}{5}$, which also satisfies $\alpha > 0$; thus,

$$\alpha^* = \frac{1}{5}.$$

(b) Assume that $p = 3\alpha - 4$, where $\alpha \geq 0$ is unknown. Compute the maximum likelihood estimate of α, denoted by $\hat{\alpha}_{ML}$, for the case when $n = 10$, and $X = 0$.
Solution: In this case, we want to find the value of α that maximizes the function

$$f(\alpha) = (5-3\alpha)^{10}$$

subject to $\alpha \geq 0$; we denote this value by α^*. The value of α that maximizes $f(\alpha)$ is clearly $\alpha = 0$, however, this would result in $p = 3 > 1$. Since

$$\frac{df(a)}{da} = -30(5-3\alpha)^9 < 0$$

for $\alpha \geq 0$, $f(\alpha)$ decreases monotonically for nonnegative values of α. Thus, we want to choose α as small as possible so that p is a valid probability; such value is $\alpha = \frac{4}{3}$, which results in $p = 0$; thus

$$\alpha^* = \frac{4}{3}.$$

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5. **[18 points]** Consider flipping a biased coin, which has $P\{\text{heads}\} = p$. Define a random variable X which is equal to the number of independent successive coin flips until 5 heads show.

(a) Determine $P\{X = 10|\text{the 3rd head shows on the 4th coin flip}\}$.
Solution: By independece of coin flips, if the 3rd head shows on the 4th coin flip, then after that 4th flip, 6 more flips are needed until the 5th head shows. Hence,

$$P\{X = 10|\text{the 3rd head shows on the 4th coin flip}\} = P\{Y = 6\} = \binom{5}{1}p^2(1-p)^4,$$

because $Y \sim$ Negative Binomial$(2, p)$.

(b) Determine the conditional probability $P\{\text{the 3rd flip shows heads}|X = 10\}$.
Solution:

$$\begin{aligned} P\{\text{the 3rd flip shows heads}|X = 10\} &= \frac{P\{\text{the 3rd flip shows heads}, X = 10\}}{P\{X = 10\}} \\ &= \frac{p^2\binom{8}{3}p^3(1-p)^5}{p\binom{9}{4}p^4(1-p)^5} = \frac{\binom{8}{3}}{\binom{9}{4}} = \frac{4}{9}. \end{aligned}$$

This is because the event $\{\text{the 3rd flip shows heads}, X = 10\}$ requires fixing two heads, each with probability p, at flips 3 and 10, then distributing the remaining 3 heads among the 8 remaining flips. Similarly, to obtain the denominator, use the fact that $X \sim$ Negative Binomial$(5, p)$. Notice that the answer is independent of p.

6. **[22 points]** The three parts are unrelated.

(a) There are two basketball teams A and B. We estimate the probability that team A beats team B (denoted by p) by having them play against each other n times. If we want to estimate p within 0.1 and with 75% confidence (using the confidence interval based on the Chebychev bound), how many games should they play?

Solution: Recall that we have

$$P\{p \in (\hat{p} - \frac{a}{2\sqrt{n}}, \hat{p} + \frac{a}{2\sqrt{n}})\} \geq 1 - \frac{1}{a^2}$$

For a 75% confidence, we need $a = 2$. Therefore, to estimate p within 0.1, we have that

$$\frac{2}{2\sqrt{n}} = 0.1,$$

or in other words, $n = 100$.

(b) For this part, assume that the probability that a basketball team is good is given by 0.7. We also know that the probability that a good team wins a game is 0.8, whereas if the team is not good the probability of winning is only 0.6. If a given basketball team plays three games and wins only two, what is the conditional probability that the team was good?

Solution: Define event $\mathcal{A}$ to denote a good *playing team*, and event $\mathcal{B}$ to winning 2 games and losing 1 (out of three games). With this notation, we are asked to compute $P(A|B)$. We can compute this quantity with Bayes and the law of total probability as follows:

$$\begin{aligned}
P(A|B) &= \frac{P(AB)}{P(B)} \\
&= \frac{P(B|A)P(A)}{P(B)} \\
&= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} \\
&= \frac{56}{83} = 0.6747,
\end{aligned}$$

where we have used

$$\begin{aligned}
P(B|A) &= 0.8^2 0.2 \\
P(B|A^c) &= 0.6^2 0.4 \\
P(A) &= 0.7 \\
P(A^c) &= 0.3
\end{aligned}$$

(c) Assume x_i for $i \in \{1, 2, \ldots, r\}$, for some integer r. Let n be another integer where $n \geq r - 1$. How many solutions are there for the equation $x_1 + x_2 + \ldots + x_r = n$ such that exactly one of the integers x_i's is zero, and all the remaining integers x_i's are positive? (Leave your answer in terms of n and r.)

Solution: We can choose the zero-valued x_i in one out of r ways. Once the zero-valued variable is removed, you are left with finding the number of positive solutions for an equation of the form $x_1 + x_2 + \ldots + x_{r-1} = n$, where we without loss of generality assumed that $x_r = 0$. There are $\binom{n+(r-1)-1}{(r-1)-1} = \binom{n+r-2}{r-2}$ such solutions. This gives a total of $r\binom{n+r-2}{r-2}$ solutions.