University of Illinois Spring 2017

ECE 313: Hour Exam I

Wednesday, March 1, 2017
8:00 p.m. — 9:15 p.m.
Last names A-S in ECEB 1002; T-Z in ECEB 1015.

1. **[3+2+1 points]** Consider three dice, with values:

   (a) $3, 3, 4, 4, 8, 8$ on the first one,
   (b) $2, 2, 6, 6, 7, 7$ on the second, and
   (c) $1, 1, 5, 5, 9, 9$ on the third one.

   One rolls all three dice simultaneously. Let A, B, C be the values on these three dice.

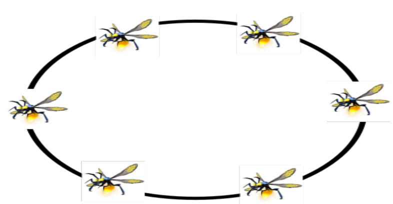
   (a) Find the probabilities $\mathbb{P}(A > B)$, $\mathbb{P}(B > C)$, $\mathbb{P}(C > A)$.
   Solution:
   $\{A > B\} = \{(3, 2), (4, 2), (8, 2), (8, 6), (8, 7)\} \Rightarrow \mathbb{P}(A > B) = 5(\frac{2}{6})^2 = 5/9$;
   $\{B > C\} = \{(2, 1), (6, 1), (7, 1), (6, 5), (7, 5)\} \Rightarrow \mathbb{P}(B > C) = 5(\frac{2}{6})^2 = 5/9$;
   $\{C > A\} = \{(5, 3), (5, 4), (9, 3), (9, 4), (9, 8)\} \Rightarrow \mathbb{P}(C > A) = 5(\frac{2}{6})^2 = 5/9$.

   (b) Find the probability $\mathbb{P}(A > B, B > C)$.
   Solution: $\{A > B, B > C\}$=\{(3,2,1),(4,2,1),(8,2,1),(8,6,1),(8,6,5),(8,7,1),(8,7,5)\}
   $P(A > B, B > C) = 7(\frac{2}{6})^3 = 7/27$

   (c) Find the probability $\mathbb{P}(A > B, B > C, C > A)$.
   Solution: $\{A > B, B > C, C > A\} = \emptyset$
   $P(A > B, B > C, C > A) = 0$

2. **[2+2 points]**

   Consider 6 fireflies, sitting in a ring. Each firefly fires independently each second with probability p.



   (a) Find the expected time until there will be exactly four fireflies alight.
   Solution: The probability for four fireflies to be alight is

$$P_1 = \binom{6}{4} p^4 (1 - p)^2$$

   So the corresponding expected time of the geometric distribution is

$$E_1 = \frac{1}{P_1} = \frac{1}{15p^4(1 - p)^2}.$$

(b) Find the probability that no two neighboring fireflies will be alight.
Solution: The total probability can be calculated by considering separately three cases with one firefly to be alight, two firefly to be alight, and three firefly to be alight, respectively, in all cases satisfying that no two neighboring fireflies are alight (note that with more than 3 fireflies alight we always have two neighboring fireflies alight). Since these events are mutually exclusive, we have that:

$$\begin{aligned} P_2 &= \mathbb{P}(1 \text{ alight and no two neighboring fireflies alight}) \\ &+\mathbb{P}(2 \text{ alight and no two neighboring fireflies alight}) \\ &+\mathbb{P}(3 \text{ alight and no two neighboring fireflies alight}) \end{aligned}$$

$$\mathbb{P}(1\,\text{alight}) = 6p(1-p)^5$$
$$\mathbb{P}(2\,\text{alight}) = 9p^2(1-p)^4$$

(there are 9 different configurations with 2 fireflies not next to each other);

$$\mathbb{P}(3\,\text{alight}) = 2p^3(1-p)^3$$

NB: The problem also admits the interpretation that no fireflies are alight; then one should add $(1-p)^6$ to the result. *Both computations accepted as correct ones.*

3. **[3+2 points]** Numbers from 00 to 99 (100 altogether) are written on a piece of paper. Each second a random number appears on the computer screen (each independent, chosen with equal probability), and is crossed out on the paper. Let's X be the time when the last is crossed out.

(a) Find the expectation of X.
Solution: Let $X = X_1 + X_2 + .. + X_{100}$ where X_i is the time needed for a new a number to appear (did not appear before). Each X_i has a geometric distribution with $p_i = \frac{101-i}{100}$. Therefore, $E[X_i] = \frac{100}{101-i}$. As a a result, we have

$$E[X] = E[X_1+X_2+..+X_{100}] = E[X_1]+E[X_2]+..+E[X_{100}] = 100\left(\frac{1}{100}+\frac{1}{99}+...+1\right).$$

(b) Find the probabilities that $X = 99, 100$.
Solution: $\mathbb{P}(X=99) = 0$
Reason : Since a number is picked each second, we need at least 100 seconds to pick 100 distinct numbers.

$\mathbb{P}(X = 100) = \frac{100!}{100^{100}}$

Reason : $X = 100$ means that all the 100 numbers are picked in 100 seconds. There are 100! ways of arranging 100 distinct numbers, each occurs with a probability $(\frac{1}{100})^{100}$.

4. **[2+3+3 points]**

Among three coins, two are fake (with probabilities of Heads and Tails being (p, q) for coin 1, (q, p) for coin 2), and one is fair (with probabilities of Heads and Tail being 1/2 each).

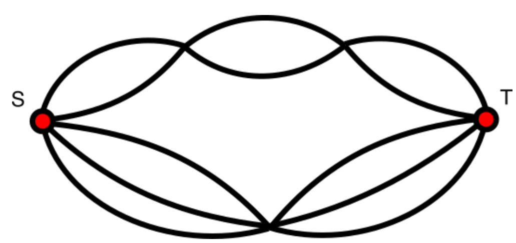
You are given one of the coins. The Hypothesis H_0 is that the coin is fair, H_1 that it is fake (in which case each of the fake coins is chosen with equal probability). You make N tosses and observe k Heads.

(a) Formulate the ML decision rule.
Solution:
$$\mathbb{P}(k|H_0) = \binom{N}{k}(\frac{1}{2})^N$$
$$\mathbb{P}(k|H_1) = \frac{1}{2}\binom{N}{k}p^k q^{N-k} + \frac{1}{2}\binom{N}{k}q^k p^{N-k}.$$
So, if
$$\Lambda(k) = \frac{\frac{1}{2}p^k q^{N-k} + \frac{1}{2}q^k p^{N-k}}{(\frac{1}{2})^N} > 1$$
H_1 is declared to be true, otherwise H_0 is declared to be true.

(b) Assume that you are told that the coin you have was chosen uniformly among the three. Formulate the MAP decision rule.
Solution: As the assumption is equivalent to having apriori probabilities $\pi_1 = 2/3, \pi_0 = 1/3$, the MAP decision rule is
$$\text{If } \Lambda(k) = \frac{\frac{1}{2}p^k q^{N-k} + \frac{1}{2}q^k p^{N-k}}{(\frac{1}{2})^N} > \frac{\pi_0}{\pi_1} = \frac{1}{2}$$
H_1 is declared to be true, otherwise H_0 is declared to be true.

(c) Given that $p = 1/3, N = 6$, find the probability that your coin is fair, if you observed $k = 1$.
Solution: According to Bayes formula,
$$\mathbb{P}(H_0|k=1) = \frac{\mathbb{P}(k=1|H_0)\pi_0}{\mathbb{P}(k=1|H_0)\pi_0 + \mathbb{P}(k=1|H_1)\pi_1},$$
that is
$$\frac{6(1/2)^6 \times 1/3}{6(1/2)^6 \times 1/3 + [(1/2)6(1/3)(2/3)^5 + (1/2)6(2/3)(1/3)^5] \times (2/3)} = 729/2905 \approx .25.$$

5. **[4 points]** Consider the network show below. The probability of the (independent) failures of each link is p.



(a) Find the probability of the network failure.
Solution: Let $q = 1 - p$.
$\mathbb{P}(\text{Network failure}) = (p^2 + p^2 + p^2 - p^2p^2 - p^2p^2 - p^2p^2 + p^2p^2p^2)(p^3 + p^3 - p^3p^3) = p^5(3 - 3p^2 + p^4)(2 - p^3)$.
Reason : $\mathbb{P}$(Network failure) $= \mathbb{P}$(upper branch fails)$\mathbb{P}$(lower branch fails). The upper branch fails if at least one of the three parallel pairs fails. The probability of each of these events is p^2, and inclusion-exclusion formula gives the first factor. The lower branch fails if at least one of the two parallel triplets fails. The probability of each of these is p^3, and inclusion-exclusion gives the second factor.

6. **[4+2 points]** Let X be the geometrically distributed random value, with probability of success p_X; Y the binomially distributed random value with parameters n, p_Y; X and Y independent.

(a) Find the probability that $X = Y + 1$. (Simplify our answer.)
Solution: $X = Y+1$ is the union of the disjoint events $\{X = k+1, Y = k\}$, $k = 0, \dots, n$. So we find

$$\mathbb{P}(X = Y+1) = \sum_{k=0}^{n} (1-p_X)^k p_X \cdot \binom{n}{k} p_Y^k (1-p_Y)^{n-k} = p_X\left((1-p_X)p_Y + (1-p_Y)\right)^n = p_X(1-p_X p_Y)^n.$$

(b) Find $\mathbb{P}(X = Y + 1 | X > Y)$. (Simplify our answer.)
Solution: We have

$$\mathbb{P}(X = Y + 1 | X > Y) = \frac{\sum_k \mathbb{P}(X=k+1, Y=k)}{\sum_k \mathbb{P}(X>k, Y=k)} = \frac{\sum_k \mathbb{P}(X=k+1)\mathbb{P}(Y=k)}{\sum_k \mathbb{P}(X>k)\mathbb{P}(Y=k)}$$

(Here we used the independence of X and Y). Further,

$$\frac{\sum_k \mathbb{P}(X=k+1)\mathbb{P}(Y=k)}{\sum_k \mathbb{P}(X>k)\mathbb{P}(Y=k)} = \frac{\sum_k \mathbb{P}(X=k+1|X>k)\mathbb{P}(X>k)\mathbb{P}(Y=k)}{\sum_k \mathbb{P}(X>k)\mathbb{P}(Y=k)} = \frac{\sum_k \mathbb{P}(X=1)\mathbb{P}(X>k)\mathbb{P}(Y=k)}{\sum_k \mathbb{P}(X>k)\mathbb{P}(Y=k)} = p_x.$$

(Here we used memoryless property of the Geometric r.v.)