More on Bernoulli trials: Improving Reliability with Triple Modular Redundancy

ECE 313

Probability with Engineering Applications

Lecture 7

Professor Ravi K. Iyer

Department of Electrical and Computer Engineering

University of Illinois

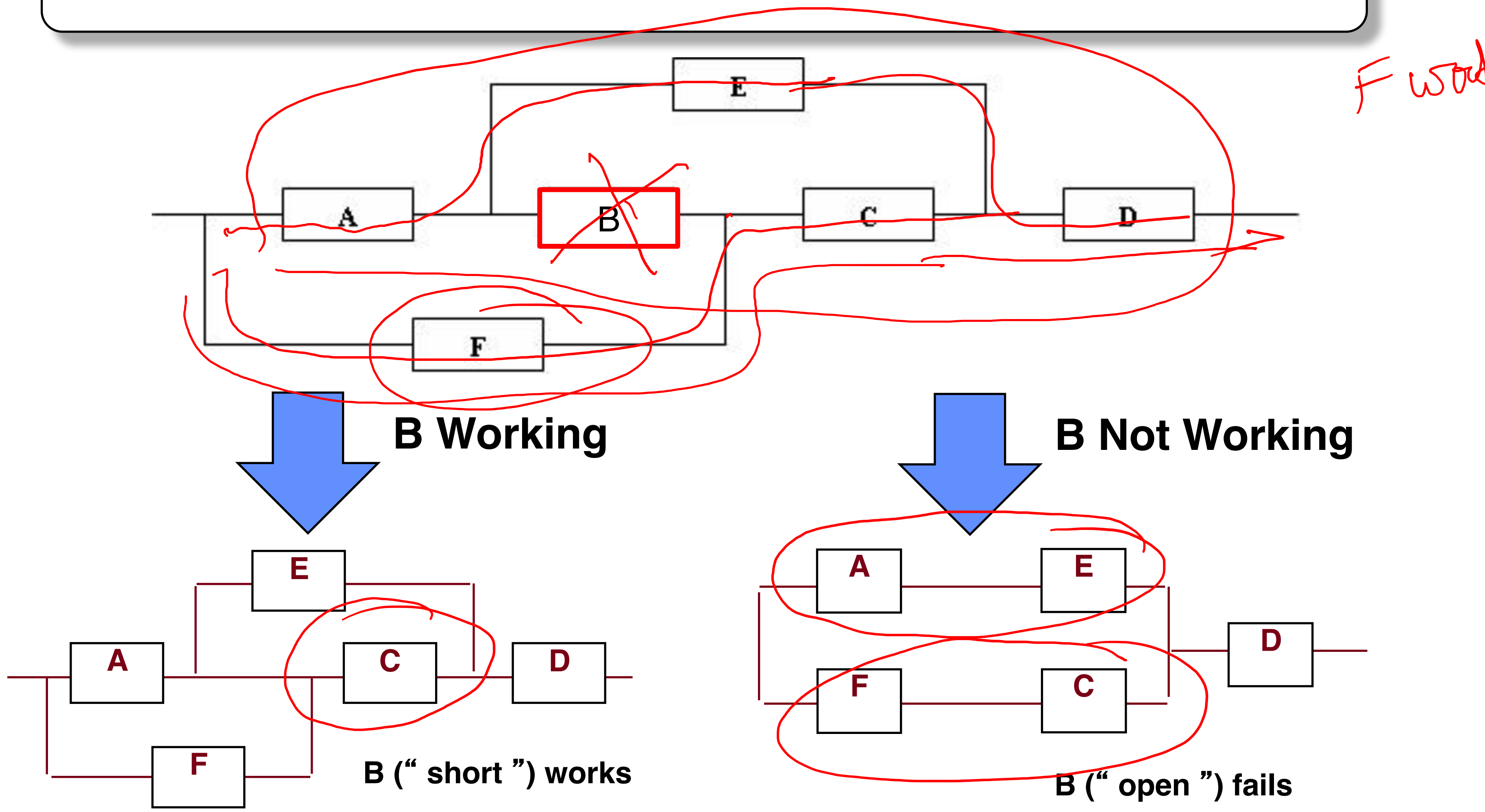
Today's Topics

- **Reliability Evaluation Applications**
  - Another non series-parallel example
  - Introducing N Modular Redundancy – Special Case of Triple Modular redundancy (TMR)
- Random Variables: Discrete and Continuous
- Prob. Mass Function (pmf), Cumulative Distribution Function (CDF)

•Announcements:

- **Homework 3 released Wednesday, Feb 15,**
- **In class group activity Wednesday as well**

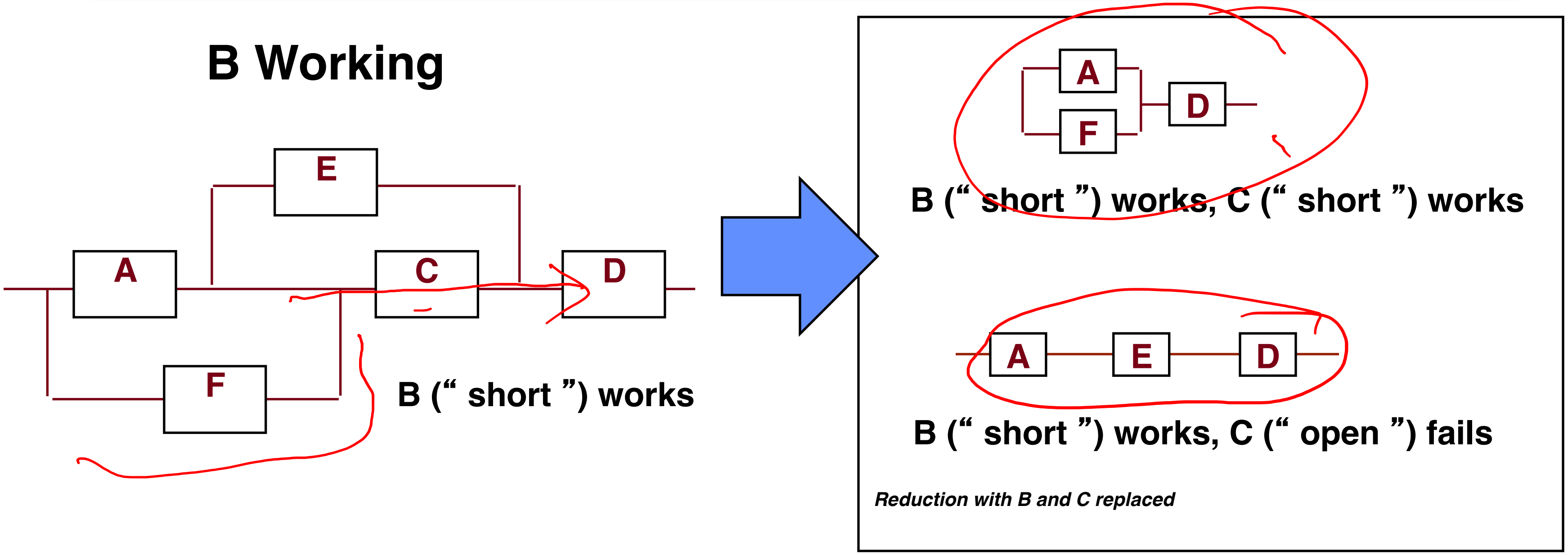
Non-Series Parallel System Example



$$R_{sys} = R_B \, P(\text{system works} \mid \text{B works}) + (1 - R_B)\,\{R_D[1 - (1 - R_A R_E)(1 - R_F R_C)]\}$$

Non-Series-Parallel-Systems (cont.)

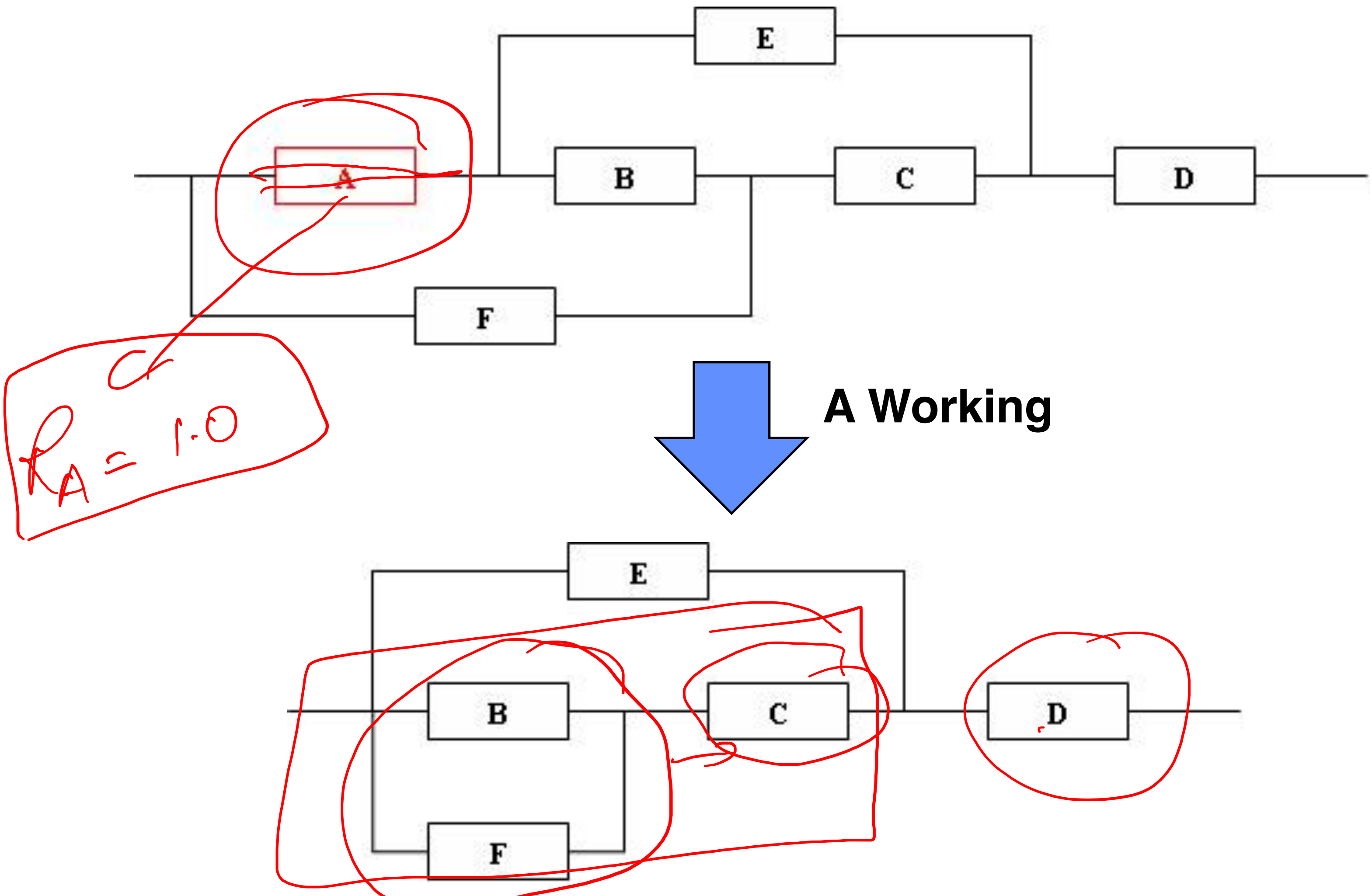
B Working



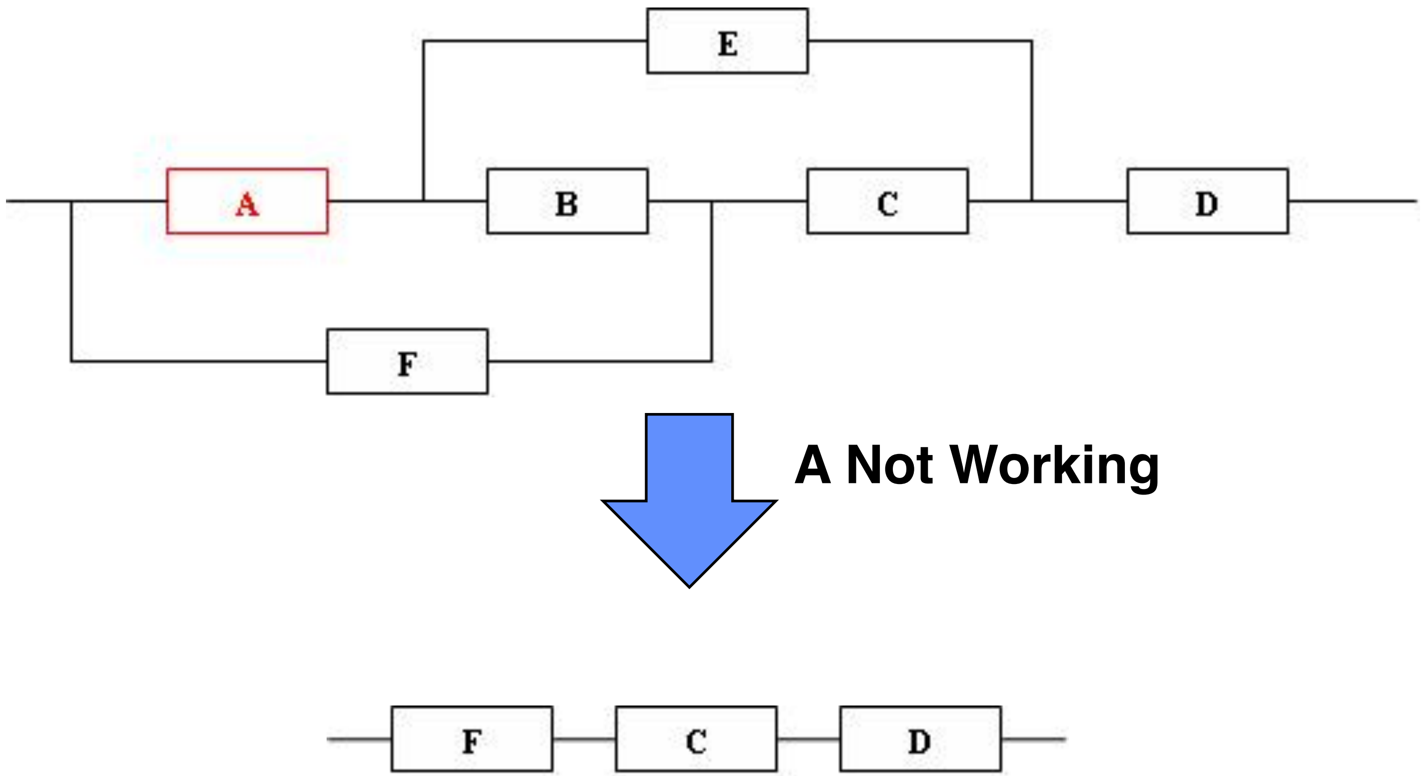
$$P(\text{system works} \mid B \text{ works}) = R_C\{R_D[1 - (1 - R_A)(1 - R_F)]\} + (1 - R_C)(R_A R_D R_E)$$

Letting $R_A \ldots R_F = R_m$ yields $\quad R_{sys} = R_m^6 - 3R_m^5 + R_m^4 + 2R_m^3$

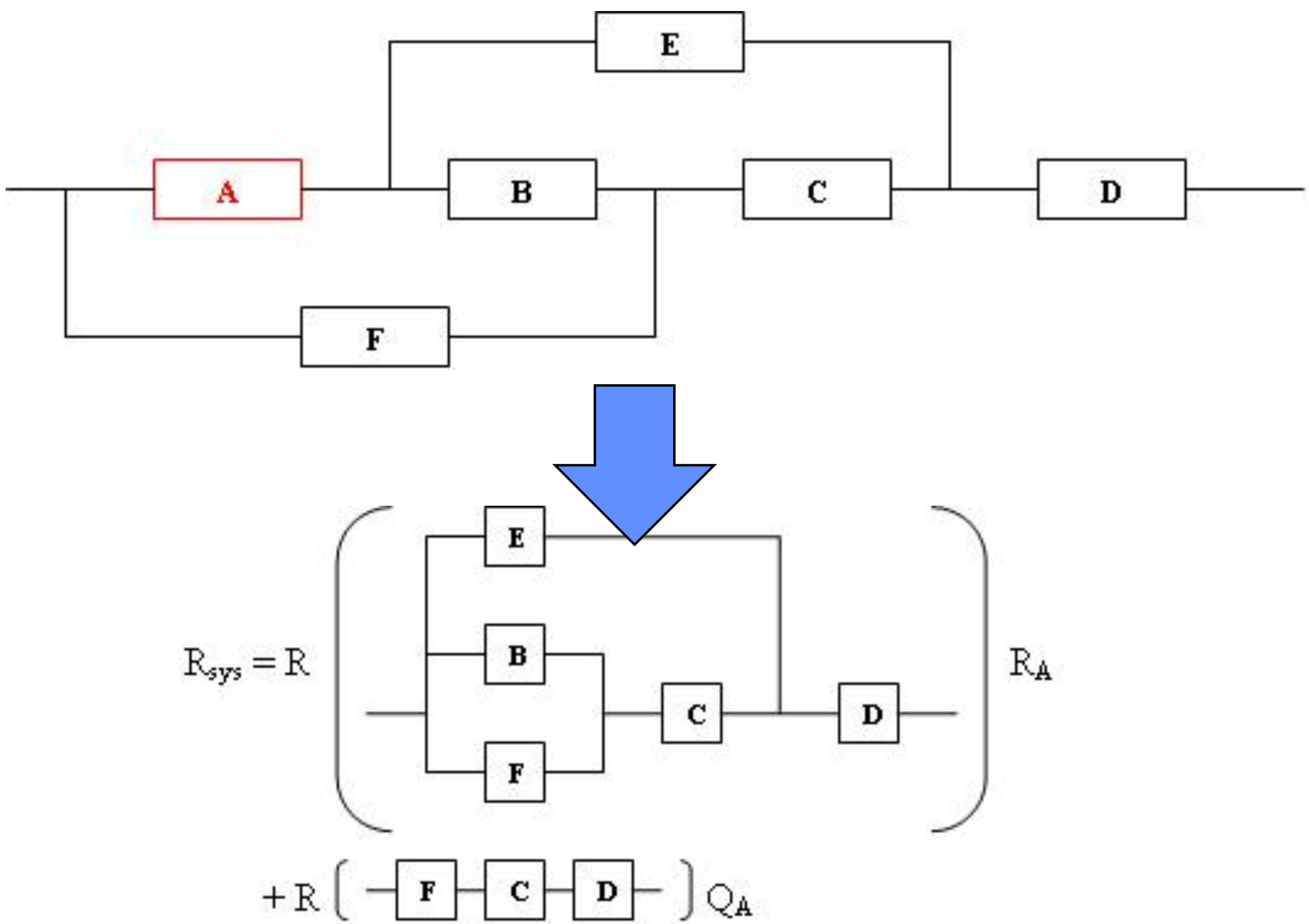
Non-Series Parallel System Example



Non-Series Parallel System Example



Non-Series Parallel System Example



Example

- Consider a binary communication channel transmitting coded words of n bits each. Assume that the probability of successful transmission of a single bit is p (and the probability of an error is q = 1-p), and the code is capable of correcting up to e (e >=0) errors.
- For example, if no coding or parity checking is used, then e = 0. If a single error correcting Hamming code is used then e =1.
- If we assume that the transmission of successive bits is independent, then the probability of successful word transmission is:

$$P_w = P(\text{"e or fewer errors in n trials"})$$
$$= \sum_{i=0}^{e} \binom{n}{i} (1-p)^i p^{n-i}$$

Sequence of Bernoulli Trials N-Modular Redundancy

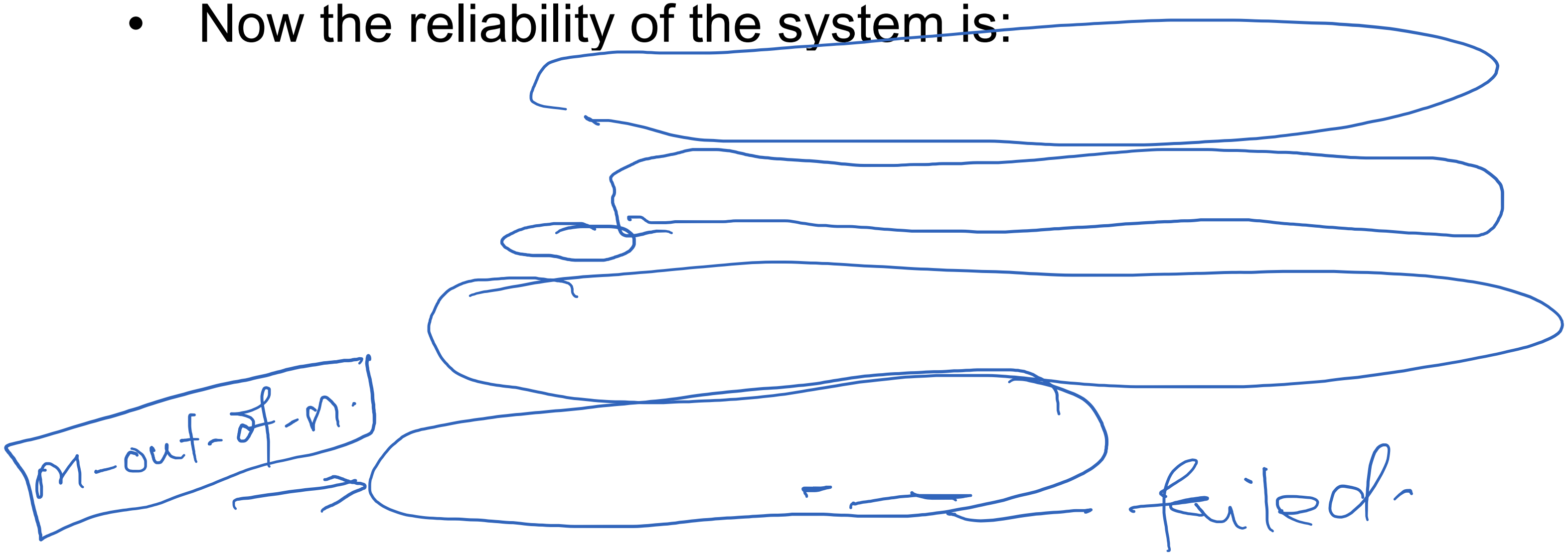
- Consider a system with n components that requires m ($\leq$n) or more components to function for the correct operation of the system (called m-out-of-n system).

- If we let m=n, then we have a series system; if we let m = 1, then we have a system with parallel redundancy.
- Assume: n components are statistically identical and function independently of each other.

- Let R denote the reliability of a component (and q = 1 – R) gives its unreliability), then the experiment of observing the status of n components can be thought of as a sequence of n Bernoulli trials with the probability of success equal R.

Bernoulli Trials Example (cont.)

- Now the reliability of the system is:

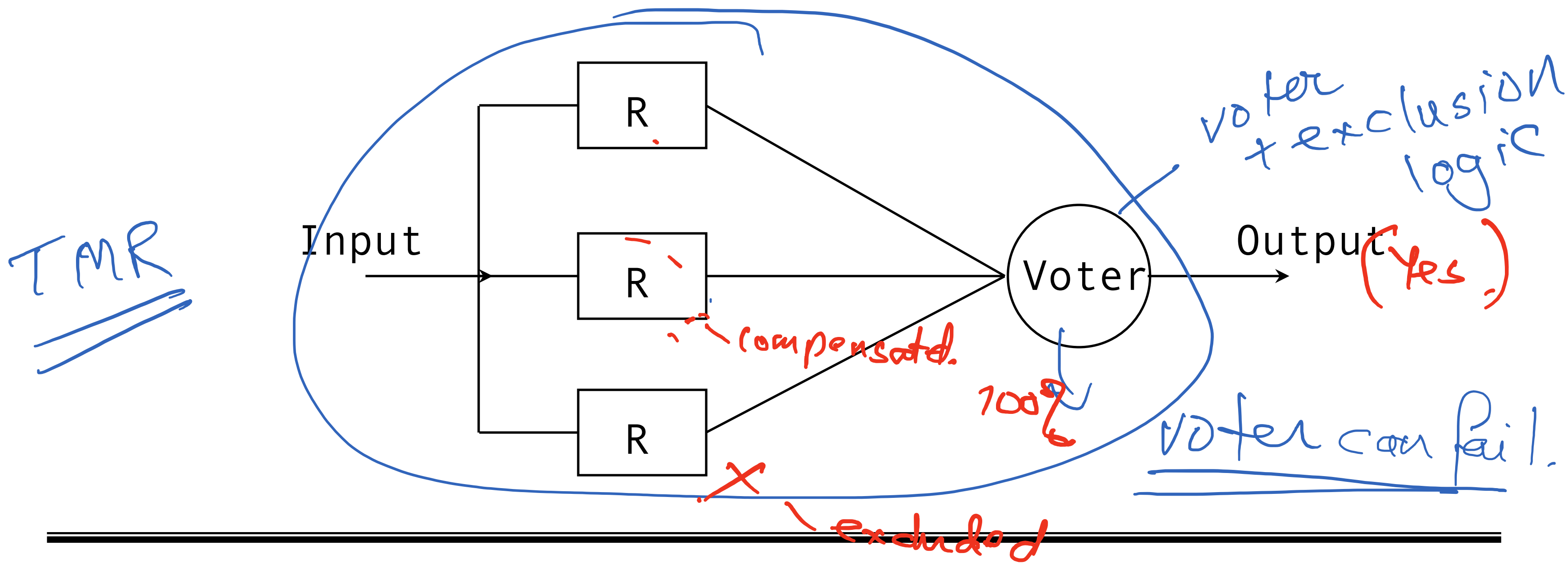


- It is easy to verify that: $R_{1|n} = R(parallel) = 1-(1-R)^n \quad and$

$$R_{n|n} = R(series) = R^n$$

Bernoulli Trials
TMR System Example

- As special case of m-out-of-n system, consider a system with triple modular redundancy (TMR) i.e. (a 3-out-of-2 system) two are required to be working for the system to function properly (i.e., n = 3 and m = 2). This is achieved by feeding the outputs of the three components into a majority voter.



Bernoulli Trials
TMR System Example (cont.)

- The reliability of TMR system is given by the expression:

- and thus

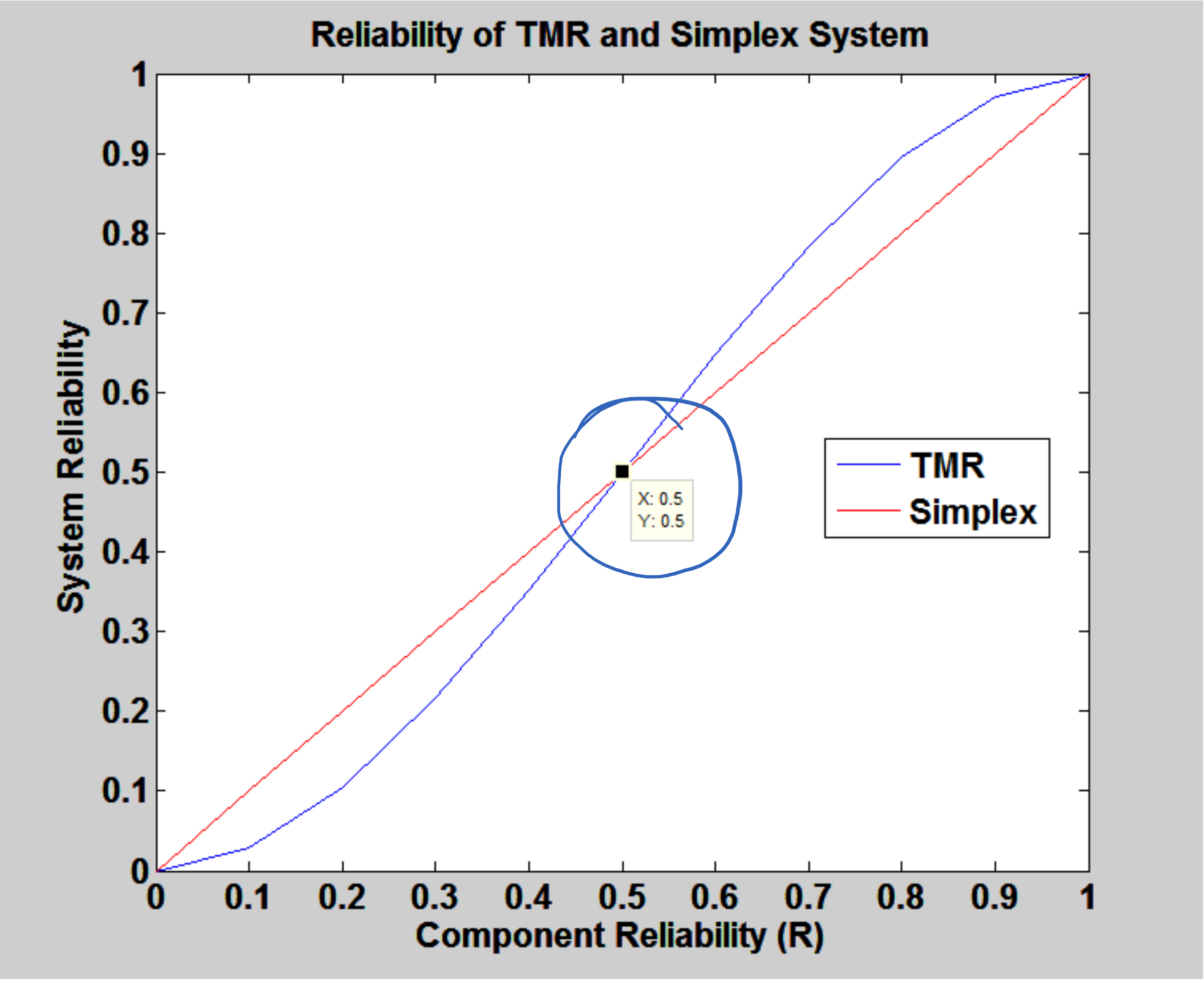
$$R_{TMR} = 3R^2 - 2R^3$$

Note that:

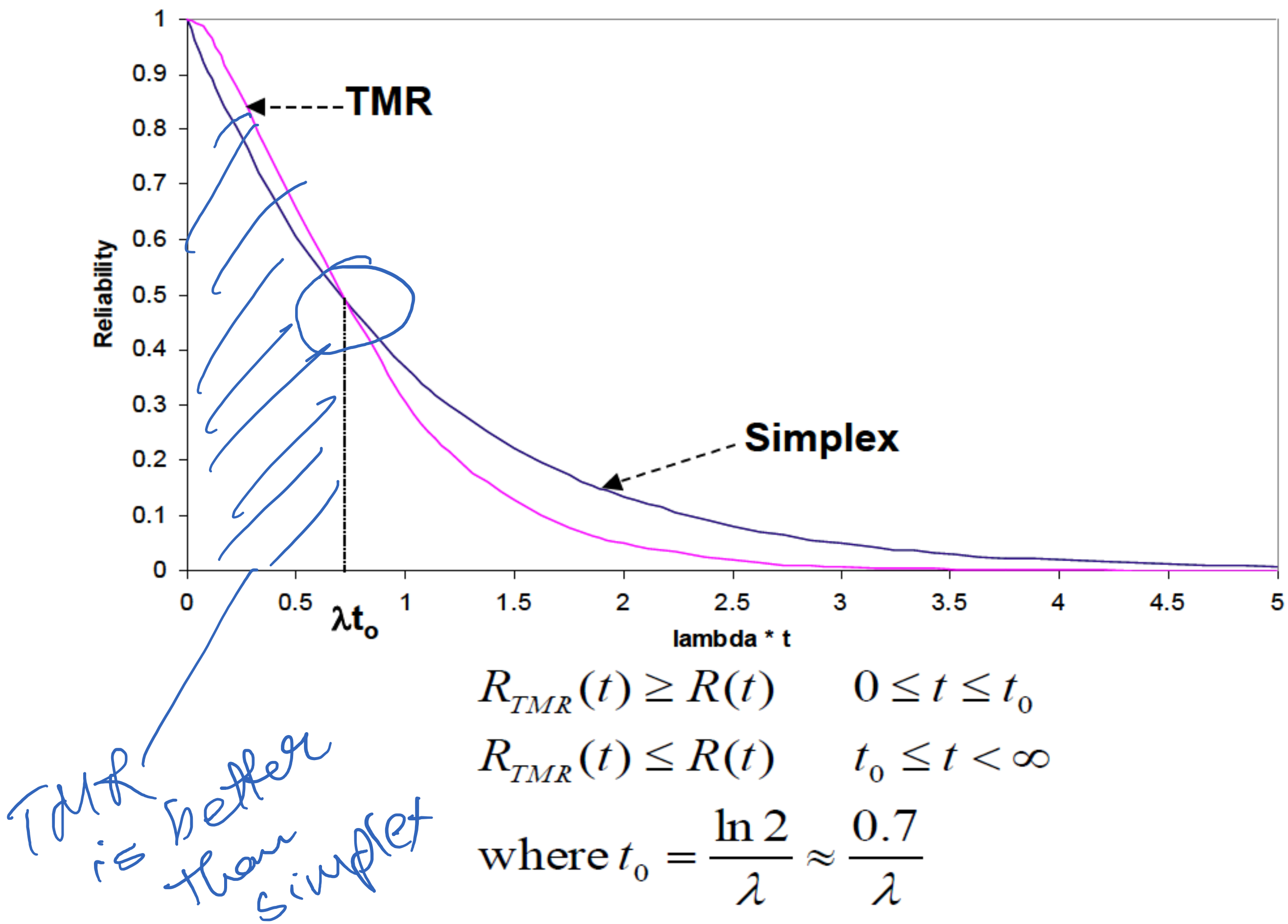
$$R_{TMR} \begin{cases} > R, & if\; R > 1/2 \\ = R, & if\; R = 1/2 \\ < R, & if\; R < 1/2 \end{cases}$$

- Thus TMR increases reliability over the simplex system only if the simplex reliability is greater than 0.5; otherwise decreases reliability
- Note: the voter output corresponds to a majority; it is possible for two or more malfunctioning units to agree on an erroneous vote.

Reliability of TMR vs. Simplex



Reliability of TMR vs. Simplex



$$R_{TMR}(t) \geq R(t) \qquad 0 \leq t \leq t_0$$

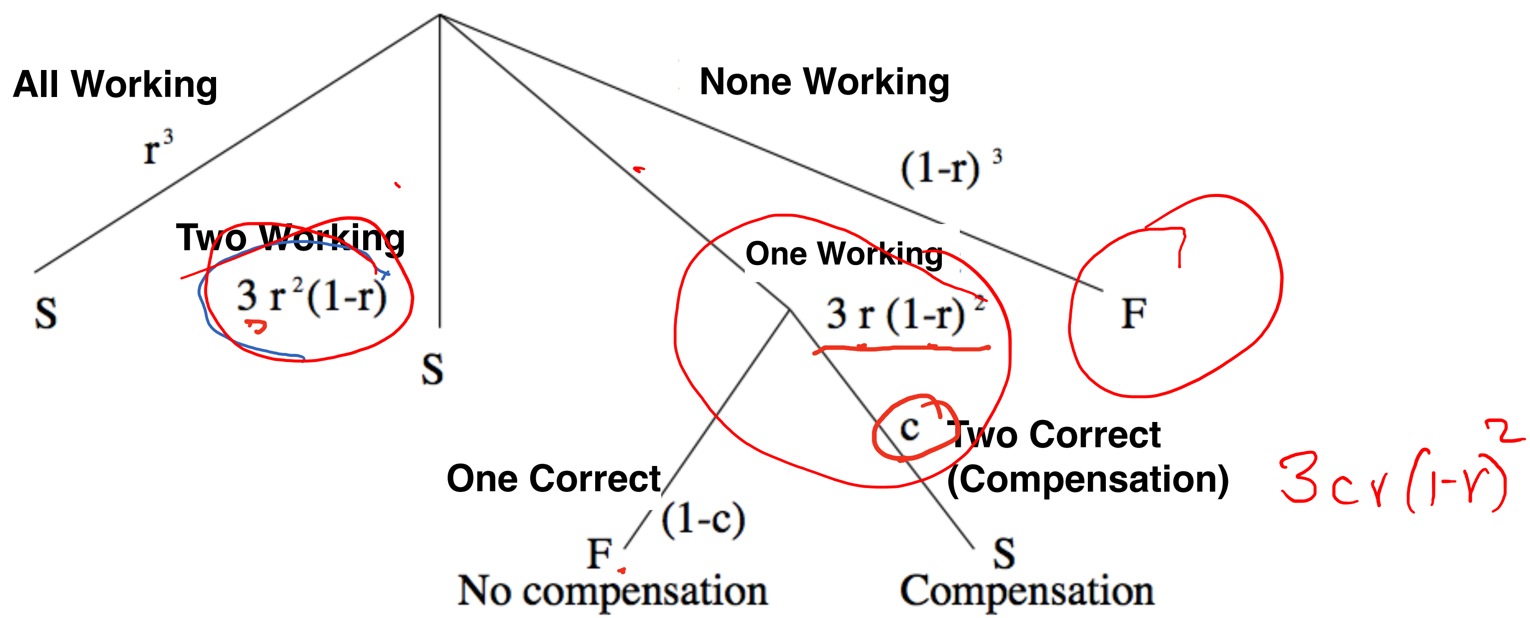
$$R_{TMR}(t) \leq R(t) \qquad t_0 \leq t < \infty$$

$$\text{where } t_0 = \frac{\ln 2}{\lambda} \approx \frac{0.7}{\lambda}$$

TMR with Compensating Error

- In the computation of TMR reliability, we assumed that when two units have failed they both produce incorrect results and, hence after voting, the wrong answer will be produced by the TMR configuration.
- In the case that two faulty units produce the opposite answers (one correct and the other incorrect) the overall result will be correct.
- Assuming that the probability of such a compensating error is *c*, derive the reliability expression for the TMR configuration.

TMR with Compensating Error



$$
\begin{aligned}
R &= r^3 + 3r^2(1-r) + 3c\,r\,(1-r)^2 \\
&= 3\,r^2 - 2\,r^3 + 3cr - 6c\,r^2 + 3c\,r^3 \\
&= 3\,r^2(1-2c) + 3cr - (2-3c)\,r^3.
\end{aligned}
$$

Random Variable

- <u>Definition:</u> Random Variable

  A random variable X on a sample space S is a function X: S $\rightarrow$ $\Re$ that assigns a real number X(s) to each sample point s $\in$ S.

<u>Example:</u> Consider a random experiment defined by a sequence of three Bernoulli trials. The sample space S consists of eight triples (where 1and 0 respectively denote success and a failure on the nth trail). The probability of successes , p, is equal 0.5.

Sample points	P(s)	X(s)
111	0.125	3
110	0.125	2
101	0.125	2
100	0.125	1
011	0.125	2
010	0.125	1
001	0.125	1
000	0.125	0

<u>Note</u> that two or more sample points might give the same value for X (i.e., X may not be a one-to-one function.), but that two different numbers in the range cannot be assigned to the same sample point (i.e., X is well defined function).

Random Variables

Function — way to map the output of a random process into Numbers

$X = \begin{cases} 1 & \text{if heads} \\ 0 & \text{if tails} \end{cases}$

commonly

Y = sum of upward face after rolling 3 dice.

$P(Y \leq 10)$ $P(Y_{even})$

Discrete/Continuous Random Variables

- The ***discrete*** random variables are either a finite or a countable number of possible values.

- Random variables that take on a continuum of possible values are known as ***continuous*** random variables.

- Example: A random variable denoting the lifetime of a car, when the car's lifetime is assumed to take on any value in some interval *(a,b) is continuous.*

Random Variable (cont.)

- *Event space*

  For a random variable X and a real number x, we define the event A_x to be the subset of S consisting of all sample points s to which the random variable X assigns the value x.

  $A_x = \{s \in S \mid X(s) = x\}$; Note that: $\bigcup_{x \in \Re} A_x = S$

  The collection of events A_x for all x defines an *event space*
- In the previous example the random variable defines four events:

  $A_0 = \{s \in S \mid X(s) = 0\} = \{(0, 0, 0)\}$

  $A_1 = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$

  $A_2 = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$

  $A_3 = \{(1, 1, 1)\}$

Discrete random variable

The random variable which is either finite or countable.

Random Variables Example 1

- Let X denote the random variable that is defined as the sum of two fair dice; then

$$P\{X=2\}=P\{(1,1)\}=\frac{1}{36},$$

$$P\{X=3\}=P\{(1,2),(2,1)\}=\frac{2}{36},$$

$$P\{X=4\}=P\{(1,3),(2,2),(3,1)\}=\frac{3}{36},$$

$$\vdots$$

$$P\{X=9\}=P\{(3,6),(4,5),(5,4),(6,3)\}=\frac{4}{36},$$

$$P\{X=10\}=P\{(4,6),(5,5),(6,4)\}=\frac{3}{36},$$

$$P\{X=11\}=P\{(5,6),(6,5)\}=\frac{2}{36},$$

$$P\{X=12\}=P\{(6,6)\}=\frac{1}{36}$$

Random Variables Example 1 (Cont'd)

- i.e., the random variable *X* can take on any integral value between two and twelve, and the probability that it takes on each value is given.

- Since *X* must take on one of the values two through twelve, we must have:

$$1 = P\left\{\bigcup_{i=2}^{12}\{X = n\}\right\} = \sum_{n=2}^{12} P\{X = n\}$$

(check from the previous equations).

Random Variables Example 2

- Suppose that our experiment consists of tossing two fair coins. Letting *Y* denote the number of heads appearing, then
- *Y* is a random variable taking on one of the values 0, 1, 2 with respective probabilities:

$$P\{Y=0\}=P\{(T,T)\}=\frac{1}{4},$$

$$P\{Y=1\}=P\{(T,H),(H,T)\}=\frac{2}{4},$$

$$P\{Y=2\}=P\{(H,H)\}=\frac{1}{4}$$

$$P\{Y=0\}+P\{Y=1\}+P\{Y=2\}=1.$$

Random Variables Example 3

- Suppose that we toss a coin until the first head appears
- Assume a probability *p* of coming up heads, on each toss.
- Letting *N (a R.V)* denote the number of flips required, and assume that the outcome of successive flips are independent,
- *N* is a random variable taking on one of the values 1, 2, 3, . . . , with respective probabilities

$$P\{N=1\}=P\{H\}=p,$$

$$P\{N=2\}=P\{(T,H)\}=(1-p)p,$$

$$P\{N=3\}=P\{(T,T,H)\}=(1-p)^2 p,$$

$$\vdots$$

$$P\{N=n\}=P\{(\underbrace{T,T,....,T}_{n-1},H)\}=(1-p)^{n-1}p, \qquad n\geq 1$$

Random Variables Example 3 (Cont'd)

- As a check, note that

$$P\left(\bigcup_{n=1}^{\infty}\{N=n\}\right)=\sum_{n=1}^{\infty}P\{N=n\}$$

$$=p\sum_{n=1}^{\infty}(1-p)^{n-1}$$

$$=\frac{p}{1-(1-p)}$$

$$=1$$

Random Variables Example 4

- Suppose that our experiment consists of seeing how long a commodity smart phone can operate before failing.
- Suppose also that we are not primarily interested in the actual lifetime of the phone but only if the phone lasts at least two years.
- We can define the random variable *I* by

$$I = \begin{cases} 1, & \text{if the lifetime of battery is two or more years} \\ 0, & \text{otherwise} \end{cases}$$

- If *E* denotes the event that the phone lasts two or more years, then the random variable *I* is known as the *indicator* random variable for event *E*. (Note that *I* equals 1 or 0 depending on whether or not *E* occurs.)

Random Variables Example 5

- Suppose that independent trials, each of which results in any of *m* possible outcomes with respective probabilities $p1, \ldots, pm$, $\sum_{i=1}^{m} p_i = 1$ are continually performed. Let *X* denote the number of trials needed until each outcome has occurred at least once.
- Rather than directly considering $P\{X = n\}$ we will first determine $P\{X > n\}$, the probability that at least one of the outcomes has not yet occurred after *n* trials. Letting *Ai* denote the event that outcome *i* has not yet occurred after the first *n* trials, $i = 1,\ldots,m$, then:

$$P\{X > n\} = P\left(\bigcup_{i=1}^{m} A_i\right)$$

$$= \sum_{i=1}^{m} P(A_i) - \sum\sum_{i<j} P(A_i A_j)$$

$$+ \sum\sum_{i<j<k}\sum P(A_i A_j A_k) - \ldots + (-1)^{m+1} P(A_1 \ldots A_m)$$

Random Variables Example 5 (Cont'd)

- Now, $P(A_i)$ is the probability that each of the first *n* trials results in a non-*i* outcome, and so by independence

$$P(A_i) = (1 - p_i)^n$$

- Similarly, $P(A_i A_j)$ is the probability that the first *n* trials all result in a non-*i* and non-*j* outcome, and so

$$P(A_i A_j) = (1 - p_i - p_j)^n$$

- As all of the other probabilities are similar, we see that

$$P\{X > n\} = \sum_{i=1}^{m} (1 - p_i)^n - \sum_{i<j}\sum (1 - p_i - p_j)^n$$
$$+ \sum_{i<j<k}\sum\sum (1 - p_i - p_j - p_k)^n - \ldots$$

Random Variables Example 5 (Cont'd)

- Since $P\{X = n\} = P\{X > n-1\} - P\{X > n\}$
- By using the algebraic identity: $(1-a)^{n-1} - (1-a)^n = a(1-a)^{n-1}$
- We see that:

$$P\{X = n\} = \sum_{i=1}^{m} p_i (1-p_i)^{n-1} - \sum\sum_{i<j} (p_i + p_j)(1 - p_i - p_j)^{n-1}$$

$$+ \sum\sum\sum_{i<j<k} (p_i + p_j + p_k)(1 - p_i - p_j - p_k)^{n-1} - \ldots$$

Discrete/Continuous Random Variables

- So far the random variables of interest were either a finite or a countable number of possible values (***discrete*** random variables).
- Random variables can also take on a continuum of possible values (known as ***continuous*** random variables).

- Example: A random variable denoting the lifetime of a car, when the car's lifetime is assumed to take on any value in some interval *(a,b).*

Discrete Random Variables: Probability Mass Function (pmf)

- A random variable that can take on at most countable number of possible values is said to be *discrete.*
- For a discrete random variable X , we define the ***probability mass function*** $p(a)$ of X by:

$$p(a) = P\{X = a\}$$

- $p(a)$ is positive for at most a countable number of values of a . i.e., if X must assume one of the values *x1, x2, …,* then

$$p(x_i) > 0, \quad i = 1,2,\ldots$$
$$p(x) = 0, \quad \text{for other values of x}$$

- Since take values *xi:*

$$\sum_{i=1}^{\infty} p(x_i) = 1$$

Cumulative Distribution Function

- The cumulative distribution function F can be expressed in terms of $p(a)$ by: $F(a) = \sum_{all\ x_i \le a} p(x_i)$
- Suppose X has a probability mass function given by

$$p(1) = \frac{1}{2},\ \ p(2) = \frac{1}{3},\ \ p(3) = \frac{1}{6}$$

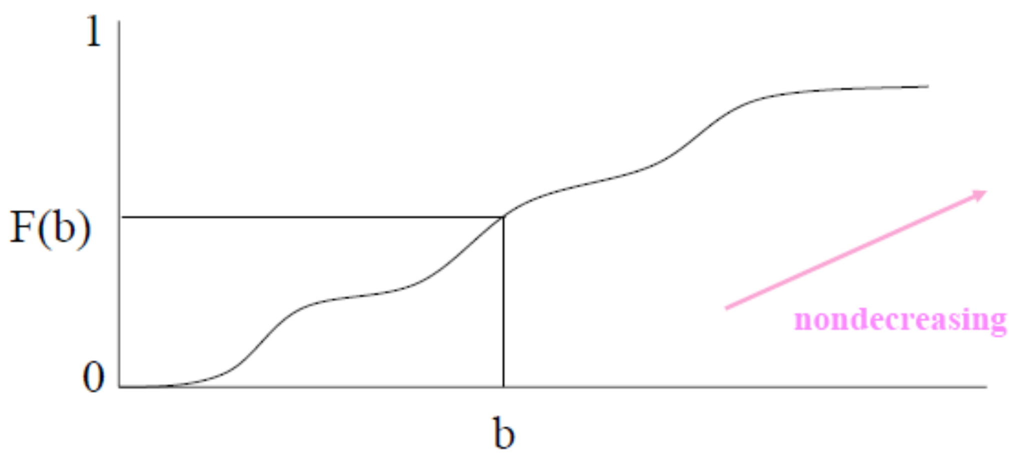
then the cumulative distribution function F of X is given by

$$F(a) = \begin{cases} 0, & a < 1 \\ \frac{1}{2}, & 1 \le a < 2 \\ \frac{5}{6}, & 2 \le a < 3 \\ 1 & 3 \le a \end{cases}$$

Cumulative Distribution Function (CDF)

- The ***cumulative distribution function* (cdf)** (or ***distribution function***) $F(\cdot)$ of a random variable X is defined for any real number $b, -\infty < b < \infty$, by $F(b) = P\{X \le b\}$

- $F(b)$ denotes the probability that the random variable X takes on a value that is less than or equal to b.



Cumulative Distribution Function (CDF)

- Some properties of cdf F are:
  i. $F(b)$ is a non-decreasing function of b,
  ii. $\lim_{b \to +\infty} F(b) = F(\infty) = 1,$
  iii. $\lim_{b \to -\infty} F(b) = F(-\infty) = 0.$

- Property *(i)* follows since for $a < b$ the event $\{X \leq a\}$ is contained in the event $\{X \leq b\}$, and so it must have a smaller probability.
- Properties *(ii)* and *(iii)* follow since X must take on some finite value.

- All probability questions about X can be answered in terms of cdf $F(\cdot)$. For example:

$$P\{a < X \leq b\} = F(b) - F(a) \qquad \text{for all } a < b$$

i.e. calculate $P\{a < X \leq b\}$ by first computing the probability that $X \leq b$ $(F(b))$ and then subtract from this the probability that $X \leq a$ $(F(a))$.

Cumulative Distribution Function

- The cumulative distribution function F can be expressed in terms of $p(a)$ by: $F(a) = \sum_{all\ x_i \le a} p(x_i)$
- Suppose X has a probability mass function given by

$$p(1) = \frac{1}{2},\ \ p(2) = \frac{1}{3},\ \ p(3) = \frac{1}{6}$$

then the cumulative distribution function F of X is given by

$$F(a) = \begin{cases} 0, & a < 1 \\ \frac{1}{2}, & 1 \le a < 2 \\ \frac{5}{6}, & 2 \le a < 3 \\ 1 & 3 \le a \end{cases}$$

Review: Discrete Random Variables

- **Discrete Random Variables:**
  - ***Probability mass function (pmf)***:

$$p(a) = P\{X = a\}$$

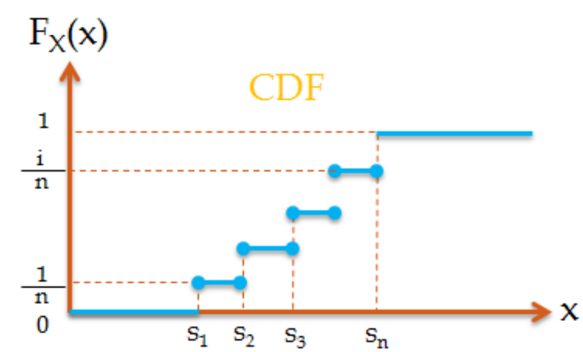
    - Properties:

$$\begin{cases} p(x_i) > 0, & i = 1,2,\ldots \\ p(x) = 0, & \text{for other values of x} \end{cases}$$

$$\sum_{i=1}^{\infty} p(x_i) = 1$$

  - **Cumulative distribution function (CDF)**:

$$F(a) = \sum_{all\ x_i \le a} p(x_i)$$



    - **A stair step function**

Discrete/Continuous Random Variables

- So far the random variables of interest were either a finite or a countable number of possible values (***discrete*** random variables).
- Random variables can also take on a continuum of possible values (known as ***continuous*** random variables).

- Example: A random variable denoting the lifetime of a car, when the car's lifetime is assumed to take on any value in some interval *(a,b).*

Continuous Random Variables

- Random variables whose set of possible values is uncountable

- X is a continuous random variable if there exists a nonnegative function $f(x)$ defined for all real $x \in (-\infty, \infty)$, having the property that for any set of B real numbers

$$P\{X \in B\} = \int_B f(x)dx$$

- $f(x)$ is called the *probability density function* of the random variable X

- The probability that X will be in B may be obtained by integrating the probability density function over the set B. Since X must assume some value, $f(x)$ must satisfy

$$1 = P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)dx$$

Continuous Random Variables Cont'd

- All probability statements about X can be answered in terms of $f(x)$
  e.g. letting $B=[a,b]$, we obtain $P\{a \leq X \leq b\} = \int_a^b f(x)dx$
- If we let $a=b$ in the preceding, then ?????
- The relationship between the cumulative distribution $F(\cdot)$ and the probability density $f(\cdot)$
- Differentiating both sides of the preceding yields

$$F(a) = P\{X \in (-\infty, a)\} = \int_{-\infty}^{a} f(x)dx$$

$$\frac{d}{da}F(a) = f(a)$$

Continuous Random Variables Cont'd

- All probability statements about X can be answered in terms of $f(x)$
  e.g. letting $B=[a,b]$, we obtain $P\{a \le X \le b\} = \int_a^b f(x)dx$
- If we let $a=b$ in the preceding, then $P\{X = a\} = \int_a^a f(x)dx = 0$
- This equation states that the probability that a continuous random variable will assume any *particular* value is zero
- The relationship between the cumulative distribution $F(\cdot)$ and the probability density $f(\cdot)$

$$F(a) = P\{X \in (-\infty, a)\} = \int_{-\infty}^{a} f(x)dx$$

- Differentiating both sides of the preceding yields

$$\frac{d}{da}F(a) = f(a)$$

Continuous Random Variables Cont'd

- That is, the density of the derivative of the cumulative distribution function.

- A somewhat more intuitive interpretation of the density function

$$P\left\{a-\frac{\varepsilon}{2}\leq X\leq a+\frac{\varepsilon}{2}\right\}=\int_{a-\varepsilon/2}^{a+\varepsilon/2}f(x)dx\approx\varepsilon f(a)$$

when ε is small

- The probability that X will be contained in an interval of length ε around the point a is approximately $\varepsilon f(a)$

Review: Continuous Random Variables

- **Continuous Random Variables:**
  - ***Probability distribution function (pdf)***:

$$P\{X \in B\} = \int_B f(x)\,dx$$

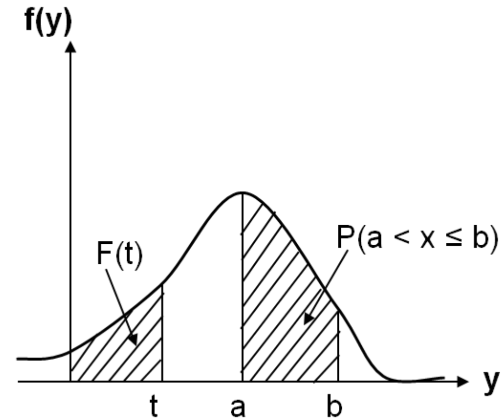
    - Properties:

$$1 = P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)dx$$

    - All probability statements about *X* can be answered by *f(x)*:

$$P\{a \le X \le b\} = \int_a^b f(x)dx$$

$$P\{X = a\} = \int_a^a f(x)dx = 0$$

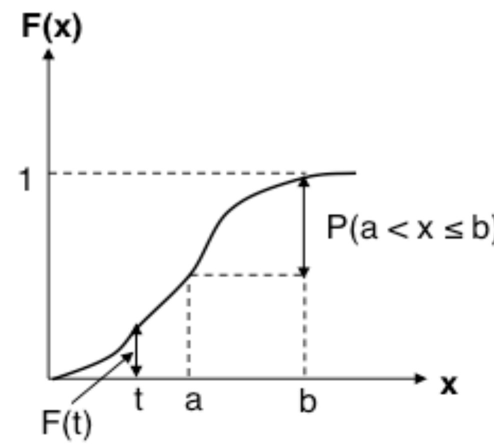


pdf

  - **Cumulative distribution function (CDF)**:

$$F_x(x) = P(X \le x) = \int_{-\infty}^{x} f_x(t)dt \;, \; -\infty < x < \infty$$

    - Properties: $\frac{d}{da}F(a) = f(a)$
    - **A continuous function**



CDF