Reliability Evaluation Applications

ECE 313

Probability with Engineering Applications

Lecture 6

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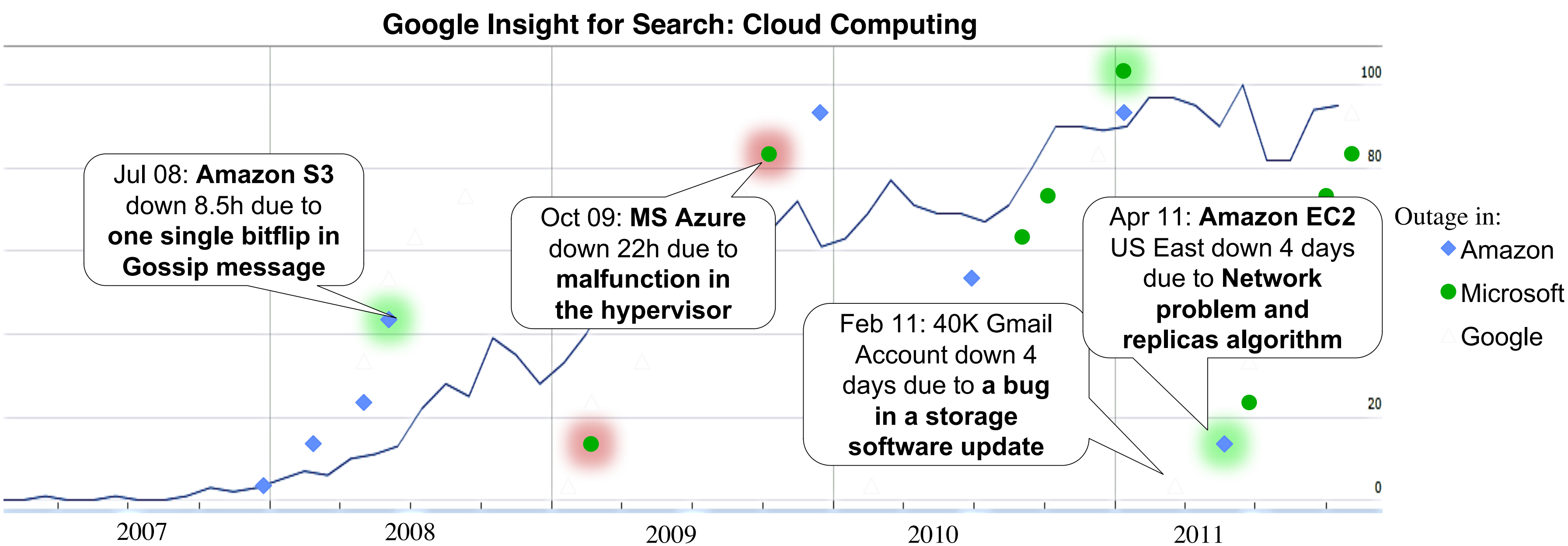
Today's Topics

- **Reliability Evaluation Applications**
  - series-parallel / non-series parallel systems

Calculating the Probability of failure: Cloud Computing Example



- Providing a higher level of reliability and availability is one of the biggest challenges of Cloud computing

R. Iyer, Z. Kalbarczyk, and N. Nakka, "Dependable Computing: Design and Assessment," Draft of forthcoming text, 2012

Application to Reliability Evaluation Series and Parallel Systems

- Consider the problem of computing reliability of so-called series-parallel systems.
- A **series system** is one in which all components are so interrelated that the entire system will fail if any one of its components fails.
- A **parallel system** is one that will fail only if all its components fail.
- We will assume that failure events of components in a system are mutually independent. Consider a series system of n components.

Application to Reliability Evaluation (cont.)

- For I=1,2,…,n, define events A_i="Component i is functioning properly." The **reliability**, R_i, of component i is defined as the probability that the component is functioning properly. Then: $R_i = P(A_i)$.
- By the assumption of series connections, the system reliability:

$$R_s = P \text{ ("The system is functioning properly.")}$$

$$=$$

$$=$$

$$=$$

Example of Effect of Complexity on Reliability

- This example demonstrates how quickly system reliability degrades with an increase in complexity.
- For example, if a system consists of five components in a series, each having a reliability of 0.970, then the system reliability is $0.970^5 = 0.859$.
- If the system complexity is increased so that it contains ten similar components, its reliability is reduced $0.970^{10} = 0.738$.
- Imagine what happens to system reliability when a large system such as a computer system consists of tens to hundreds of thousands of components.

Increasing Reliability Using Redundancy

- One way to increase the reliability of a system is to use redundancy, i.e., to replicate components with small reliabilities. This is called **parallel redundancy**.
- Consider a system consisting of *n* independent components in parallel; the system fails to function only if *n* components have failed.
- Define event A_i="The component *i* is functioning properly" and

  A_p="The parallel system of *n* components is functioning properly."

  Also let $R_i=P(A_i)$ and $R_p=P(A_p)$.

  To establish a relation between A_p and the A_i's, it is easier to consider the complementary events.

  Thus: $\overline{A}_p$ = "The parallel system has failed."

  = "All *n* components have failed."

  =

- Therefore, by independence:

$$P(\overline{A}_p) = P(\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_n) =$$

Increasing Reliability Using Redundancy (cont.)

- Now let $F_p=1-R_p$ be the unreliability of the parallel system, and similarly let $F_i=1-R_i$ be the unreliability of component *i*.
- Then, since A_i and $\overline{A}_i$ are mutually exclusive and collectively exhaustive events, we have: $1 = P(S) = P(A_i) + P(\overline{A}_i)$
- And: $F_i = P(\overline{A}_i) = 1 - P(A_i)$
- Then: $F_p = P(\overline{A}_i) = \prod_{i-1}^{n} F_i$
- And: $R_p = 1 - F_p = 1 - \prod_{i=1}^{n} (1 - R_i)$

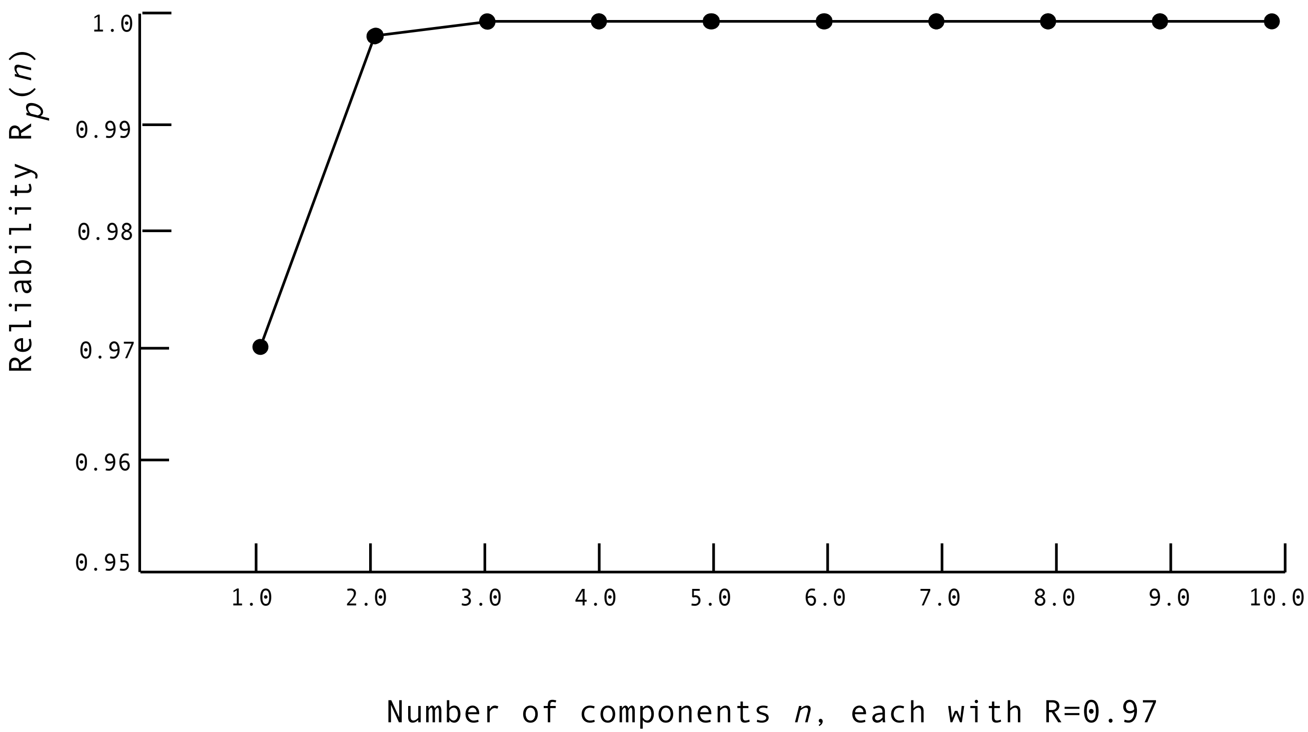
Product Law of Unreliabilities

- Thus, for parallel systems of *n* independent components, we have a **product law of unreliabilities** analogous to the **product law of reliabilities** of series systems.
- If we have a parallel system of five components, each with a reliability of 0.970, then the system reliability is increased to:

$$1-(1-0.970)^5 = 1-(0.03)^5 = 1-0.0000000243 = 0.9999999757$$

- However, we should be aware of the **law of diminishing returns**.

Reliability Curve of a Parallel Redundant System



Reliability of Series-Parallel Systems

- We can use formulas parallel and series systems in combination to compute the reliability of a system having both series and parallel parts (**series-parallel systems**).

- Consider a series-parallel system of n serial stages, where stage *i* consists of n_i identical components in parallel.

Reliability of Series-Parallel Systems (cont.)

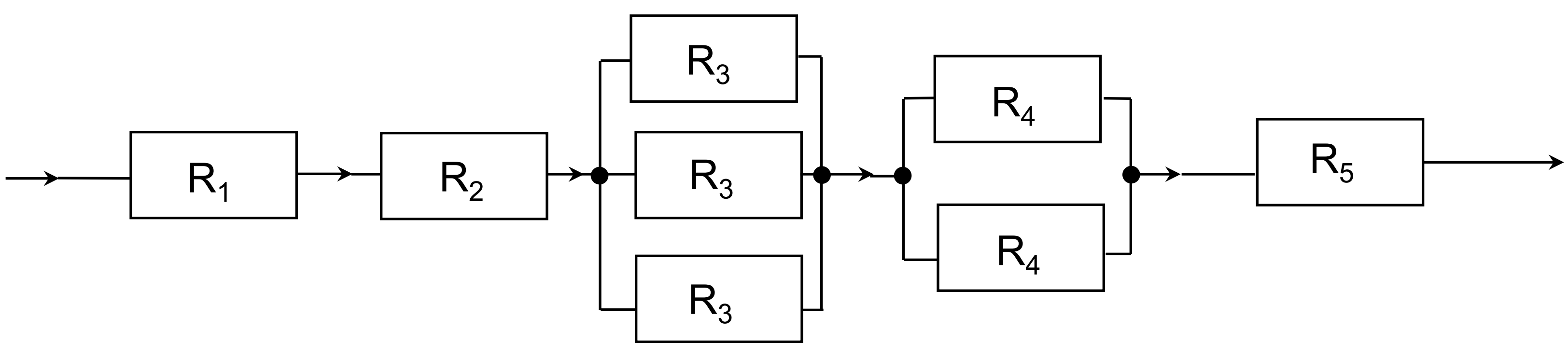
- The reliability of each component of stage *i* is R_i.
- Assuming that all components are independent, R_{sp} :

$$R_{sp} = \prod_{i=1}^{n} [1 - (1 - R_i)^{n_i}]$$

Series-Parallel System Example

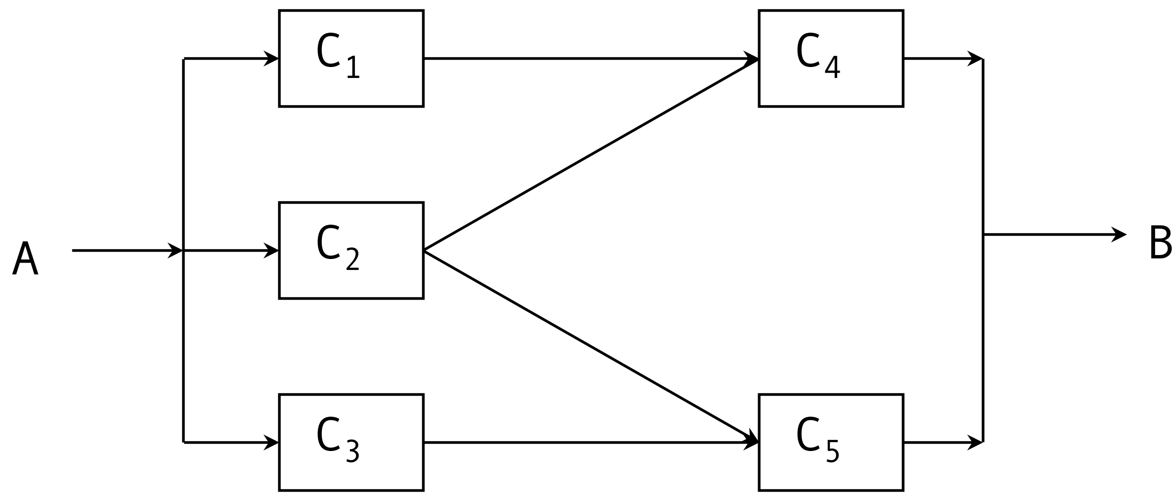
- Consider the series-parallel system shown in the diagram below, consisting of five stages, with $n_1=n_2=n_5=1$, $n_3=3$, $n_4=2$, and $R_1=0.95$, $R_2=0.99$, $R_3=0.70$, $R_4=0.75$, and $R_5=0.9$.

- Then: $R_{sp} = 0.95 \cdot 0.99 \cdot (1-0.3^3) \cdot (1-0.25^2) \cdot 0.9 = 0.772$



Application of Bayes
A More Complex System: Example

- Consider evaluating the reliability R of the five-component system. The system is said to be functioning properly only if all the components on at least one path from point A to point B are functioning properly.



Bayes' Formula Example 3 (cont.)

- Define for i = 1, 2, …, 5 event X_i = "Component i is functioning properly"
  - let R_i = reliability of component i = $P(X_i)$
  - let X = "System is functioning properly
  - let R = system reliability = P(X)
- Thus X is a union of four events

$$X = (X_1 \cap X_4) \cup (X_2 \cap X_4) \cup (X_2 \cap X_5) \cup (X_3 \cap X_5)$$

- These four events are not mutually exclusive. Therefore, we cannot directly use axiom (A3). Note, however, that we could use relation (Rd), which does apply to union of interesting events. But this is computationally tedious for a relatively long list of events. Instead, use **the theorem of total probability**, we have:

$$P(X) = P(X \cap X_2) + P(X \cap \overline{X_2})$$

$$P(X) = P(X \mid X_2)P(X_2) + P(X \mid \overline{X}_2)P(\overline{X}_2) = P(X \mid X_2)R_2 + P(X \mid \overline{X}_2)(1 - R_2)$$

Bayes' Formula Example 3 (cont.)

- Now to compute $P(X|X_2)$observe that since component C_2 is functioning, the status of components C_1 and C_3 is irrelevant.

- To compute $P(X|\overline{X}_2)$, since C_2 is known to have failed, the

Next Topics

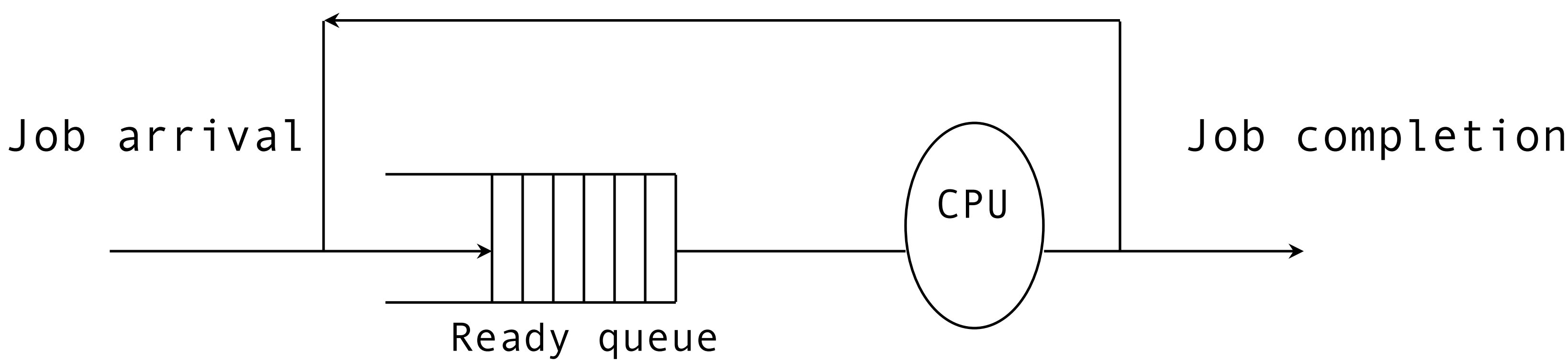
- Bernoulli Trials
- TMR Systems

Bernoulli Trials

- Physical situations of interest:
- **1.** Observe n consecutive executions of an **if** statement, with success = "**then** clause is executed" and failure ="**else** clause is executed"
- 2. Examine components produced on an assembly line, with success = "acceptable" and failure = "defective"
- 3. Transmit binary digits through a communication channel, with success = "digit received correctly" and failure = "digit received incorrectly"

Bernoulli Trials (cont.)

4. Consider a time-sharing computer system that allocates a finite quantum (or time slice) to a job scheduled for processor service. Observe n time-slice terminations, with success = "job has completed processing" and failure = "job still requires processing and joins the tail end of the ready queue of processes



Bernoulli Trials (Cont'd)

- Consider a random experiment that has two possible outcomes,. Let the probabilities of the two outcomes be p and q, respectively, with $p + q = 1$.

- Now consider the compound experiment: A sequence of n independent repetitions of this experiment. Such a sequence: is known as a **sequence of Bernoulli trials**.

Bernoulli Trials (cont.)

- Let 0 denote failure and 1 denote success. Let S_n be the sample space of an experiment involving n Bernoulli trials, defined by:

- The probability assignment over the sample space S_1 is already specified: $P(0) = q \leq 0$, $P(1) = p \leq 0$, and $p + q = 1$. We wish to assign probabilities to the points in S_n.

- Let A_i = “Success on trial i” and $\overline{A}_i$ = “Failure on trial i” then $P(A_i) = p$ and $P(A_i) = \overline{q}$.

Bernoulli Trials (cont.)

- Consider s an element of S_n such that s = (1, 1, …, 1, 0, 0, …, 0) [k 1's and (n-k) 0's]. Then the elementary event {s} can be written:

Bernoulli Trials (cont.)

- Therefore:

- Similarly, any sample point with k 1's and (n-k) 0's is assigned probability $p^k q^{n-k}$. Noting that there are $\binom{n}{k}$ such points, the probability of obtaining exactly k successes in n trials is :

- Verify that expression for P(s) is a legitimate probability assignment over the sample space Sn since

  by the binomial theorem.

Example

- Consider a binary communication channel transmitting coded words of n bits each. Assume that the probability of successful transmission of a single bit is p (and the probability of an error is q = 1-p), and the code is capable of correcting up to e (e >=0) errors.
- For example, if no coding or parity checking is used, then e = 0. If a single error correcting Hamming code is used then e =1.
- If we assume that the transmission of successive bits is independent, then the probability of successful word transmission is:

$$P_w = P(\text{"e or fewer errors in n trials"})$$
$$= \sum_{i=0}^{e} \binom{n}{i} (1-p)^i p^{n-i}$$

Bernoulli Trials Example

- Consider a system with n components that requires m ($\leq$n) or more components to function for the correct operation of the system (called m-out-of-n system).

- If we let m=n, then we have a series system; if we let m = 1, then we have a system with parallel redundancy.
- Assume: n components are statistically identical and function independently of each other.

- Let R denote the reliability of a component (and q = 1 - R gives its unreliability), then the experiment of observing the status of n components can be thought of as a sequence of n Bernoulli trials with the probability of success equal R.

Bernoulli Trials Example (cont.)

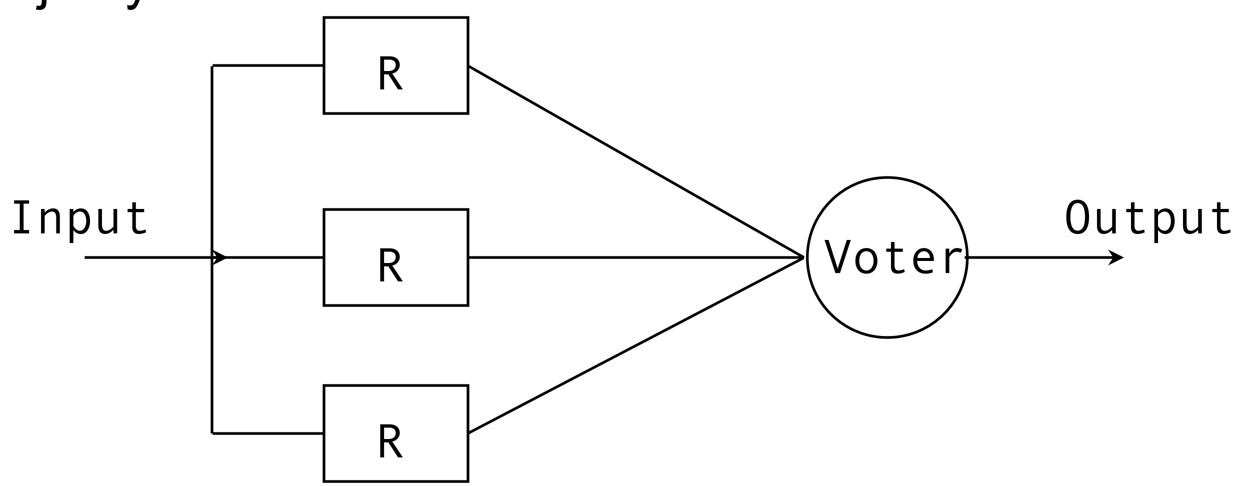
- Now the reliability of the system is:

- It is easy to verify that: $R_{1|n} = R(parallel) = 1 - (1 - R)^n \quad and$

$$R_{n|n} = R(series) = R^n$$

Bernoulli Trials
TMR System Example

- As special case of m-out-of-n system, consider a system with triple modular redundancy (TMR). In such a system there are three components, two of which are required to be in working order for the system to function properly (i.e., n = 3 and m = 2). This is achieved by feeding the outputs of the three components into a majority voter.



Bernoulli Trials
TMR System Example (cont.)

- The reliability of TMR system is given by the expression:

- and thus

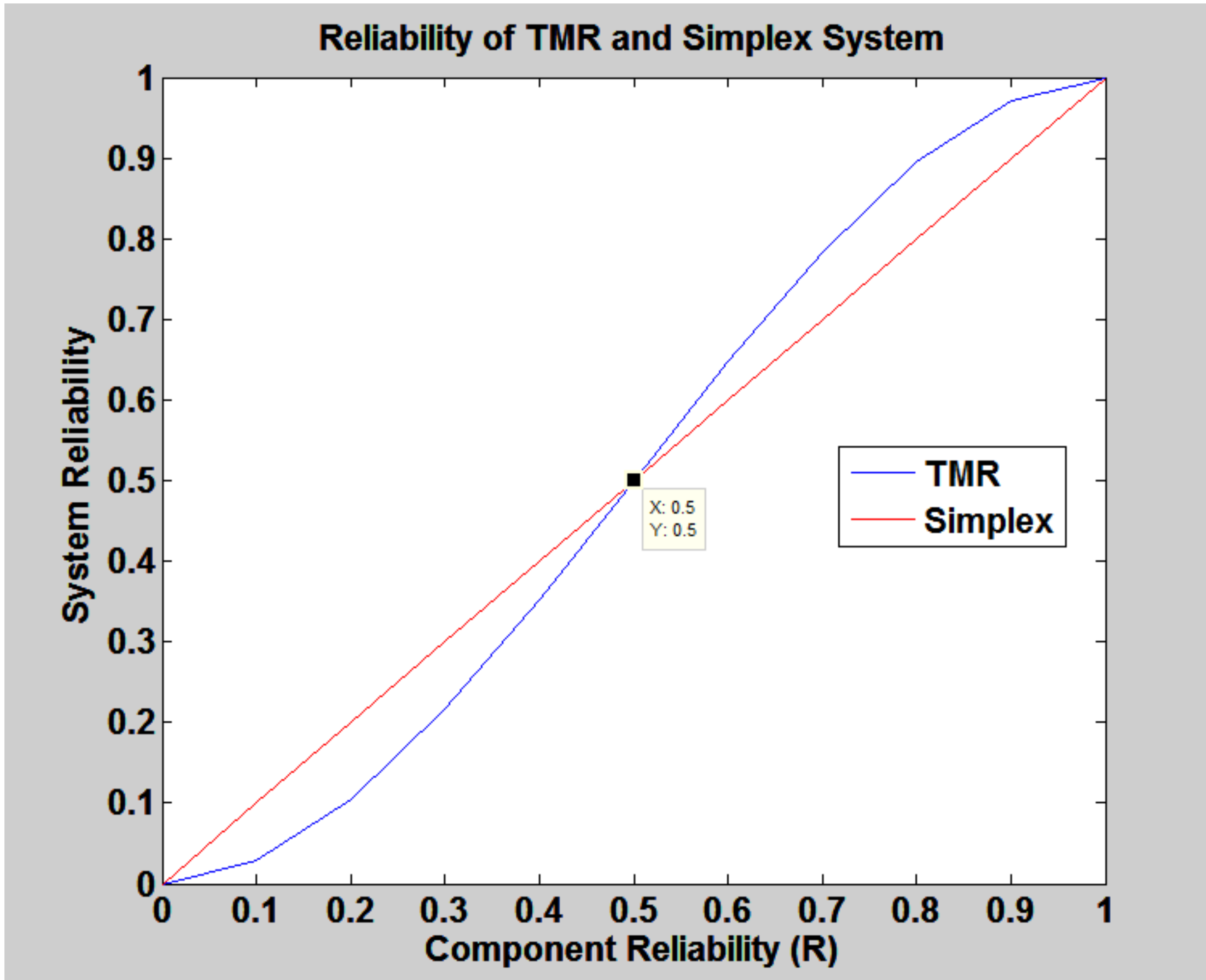
$$R_{TMR} = 3R^2 - 2R^3$$

Note that:

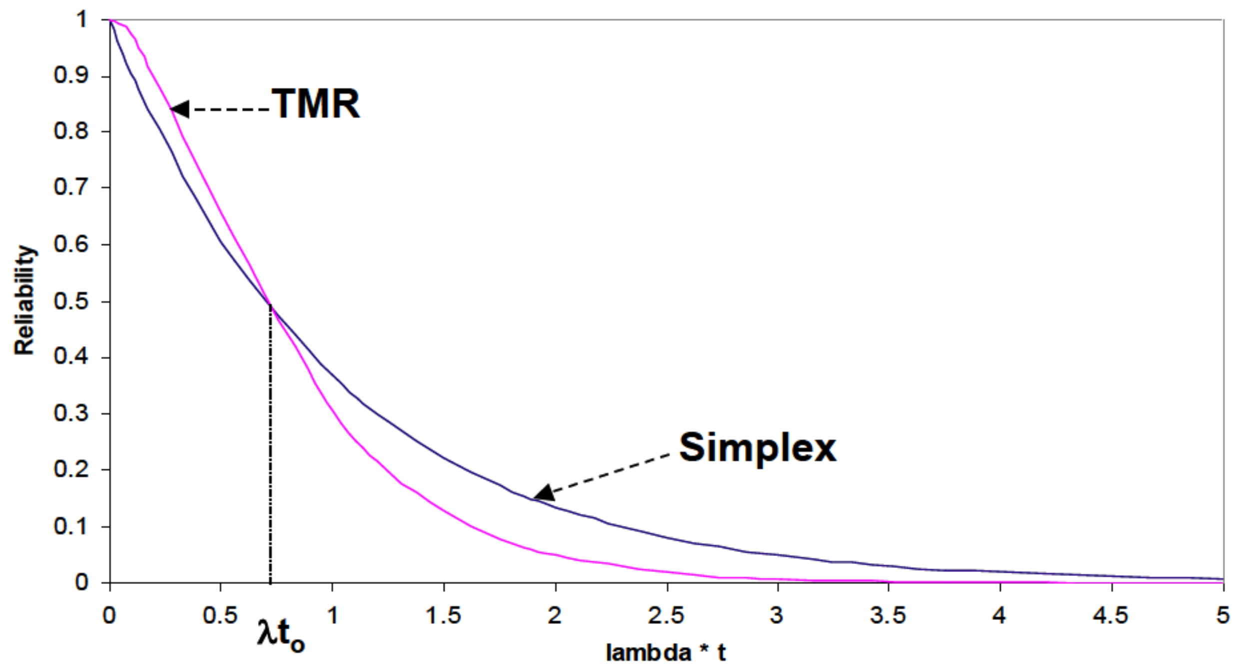
$$R_{TMR} \begin{cases} > R, & if\, R > \frac{1}{2} \\ = R, & if\, R = \frac{1}{2} \\ < R, & if\, R < \frac{1}{2} \end{cases}$$

- Thus TMR increases reliability over the simplex system only if the simplex reliability is greater than 0.5; otherwise decreases reliability
- Note: the voter output corresponds to a majority; it is possible for two or more malfunctioning units to agree on an erroneous vote.

Reliability of TMR vs. Simplex



Reliability of TMR vs. Simplex



$$R_{TMR}(t) \geq R(t) \qquad 0 \leq t \leq t_0$$

$$R_{TMR}(t) \leq R(t) \qquad t_0 \leq t < \infty$$

$$\text{where } t_0 = \frac{\ln 2}{\lambda} \approx \frac{0.7}{\lambda}$$

Random Variable

- <u>Definition:</u> Random Variable

  A random variable X on a sample space S is a function X: S $\rightarrow$ $\Re$ that assigns a real number X(s) to each sample point s $\in$ S.

  <u>Example:</u> Consider a random experiment defined by a sequence of three Bernoulli trials. The sample space S consists of eight triples (where 1and 0 respectively denote success and a failure on the nth trail). The probability of successes , p, is equal 0.5.

Sample points	P(s)	X(s)
111	0.125	3
110	0.125	2
101	0.125	2
100	0.125	1
011	0.125	2
010	0.125	1
001	0.125	1
000	0.125	0

<u>Note</u> that two or more sample points might give the same value for X (i.e., X may not be a one-to-one function.), but that two different numbers in the range cannot be assigned to the same sample point (i.e., X is well defined function).

Random Variable (cont.)

- *Event space*

  For a random variable X and a real number x, we define the event A_x to be the subset of S consisting of all sample points s to which the random variable X assigns the value x.

  $A_x = \{s \in S \mid X(s) = x\}$; Note that: $\bigcup_{x \in \Re} A_x = S$

  The collection of events A_x for all x defines an *event space*
- In the previous example the random variable defines four events:

  $A_0 = \{s \in S \mid X(s) = 0\} = \{(0, 0, 0)\}$

  $A_1 = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$

  $A_2 = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$

  $A_3 = \{(1, 1, 1)\}$

Discrete random variable

The random variable which is either finite or countable.

Probability Mass Function

- *Probability mass function (pmf)* or the *discrete density function* of the random variable X $p_X(x)$ gives:

  the probability that the value of the random variable X obtained on a performance of the experiment is equal to x.

$$p_X(x) = P(X = x) = \sum_{X(s)=x} P(s$$

- Properties of the pmf:

  (p1) $0 \leq p_X(x) \leq 1$ for all x Œ← ; (since $p_X(x)$ is a probability)

  (p2) $\sum_{x \in \Re} p_X(x) = 1$ (since the random variable assigns some value x Œ← to each sample point s Œ S)

  (p3) for a discrete random variable X, the set $\{x | p_X(x) \neq 0\}$ is a finite or countably infinite subset of real numbers. Let denote this set by $\{x_1, x_2, ...\}$. Then the property (p2) can be restated as: $\sum_i p_X(x_i) = 1$

Probability Mass Function
Examples

- For the previous example we can easily obtain $p_X(x)$

$$p_X(0) = 0.125$$
$$p_X(1) = 0.375$$
$$p_X(2) = 0.375$$
$$p_X(3) = 0.125$$

- For the example of a computer system with five tape drives, and defining the random variable X = "the number of available tape drives" we have:
- $p_X(0) = 1/32$, $p_X(1) = 5/32$, $p_X(2) = 10/32$
  $p_X(3) = 10/32$, $p_X(4) = 5/32$, $p_X(5) = 1/32$