Conditional Probability

ECE 313

Probability with Engineering Applications

Lecture 3

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Today's Topics and Announcements

- Conditional Probability
- Bayes Rule
- Theorem of Total Probability
- Bayes Formula
- Independence of Events

Today's Topics and Announcements

- **Announcements:**
  - **Homework 1** is **due this Feb 1st**
  - **Mini Project 1** will be **posted Feb 1st**
  - Midterm will be in class on March 15th
  - Please visit **Piazza** for asking questions and recent updates

Conditional Probability

- Probability of an Event A given that the outcome *s* is in a subset B (or Event B).
- Called conditional probability of Event A given that Event B has occurred.

  Conditional probability A given B: (P(A|B))

- Given B has occurred, the sample point (element) corresponding to this conditional probability must be in B and not in $\overline{B}$.
- Define

$$P(s|B) = \begin{cases} \frac{P(s)}{P(B)} & \text{if } s \in B \\ 0 & \text{if } s \in Bc \end{cases}$$

(original probability of a sample point is scaled by $1/P(B)$ so that the probability of all sample points in B add up to 1)

Conditional Probability (cont.)

$$P(A \mid B) = \sum_{s \in A} P(s \mid B)$$

noting that

$$A = (A \cap B) \cup (A \cap \overline{B})$$

$$P(A \mid B) = \sum_{s \in A \cap \overline{B}} P(s \mid B) + \sum_{s \in A \cap B} P(s \mid B)$$

$$= \sum_{s \in A \cap B} P(s) / P(B)$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Conditional Probability

- *Conditional probability of A given B (P(A|B))* defines the conditional probability of the event A given that the event B occurs and is given by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

  if P(B) $\neq$ 0 and is undefined otherwise.
- A rearrangement of the above definition gives the following *multiplication rule (MR)*

$$P(A \cap B) = \begin{cases} P(B)P(A|B) & \text{if } P(B) \neq 0 \\ P(A)P(B|A) & \text{if } P(A) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Example: Cloud Computing SaaS

- There are 275,790 files submitted over the period of 3 months to a SW platform
- Suppose in your data set you have 5,000 failure entries, including:
  - 1,200 User Data Failures
  - 3,800 Platform Failures

- What is the probability of a User Data Failure? *(Unconditional Probability)*
  - P(User Data Failure) = 1,200/275,790 = 0.4%

- If a failure happens, what is the probability of that failure being a User Data Failure? *(Conditional Probability)*
  - P(User Data Failure | Failure Occurred) = 1200/5000 = 24%

Theorem of Total Probability

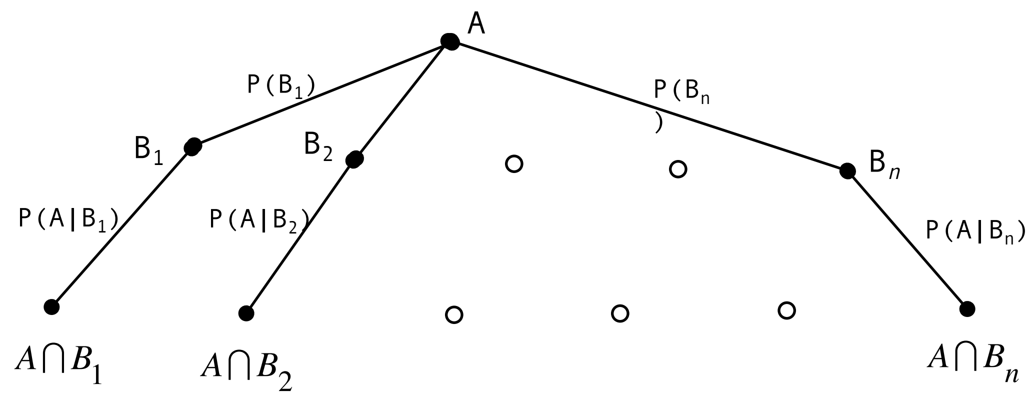
- An event B (probability P(B)) partitions a sample space S into two disjoint subsets B and $\overline{B}$ (exhaustive and exclusive)
- Consider S' ={B,$\overline{B}$} with associated probabilities P(B) and P($\overline{B}$), S' (**event space**).
- If A is another event in S', then: $A = (A \cap B) \cup (A \cap \overline{B})$.
- Then: $P(A) = P(A \cap B) + P(A \cap \overline{B})$
- And, using the definition of **conditional probability**, this equals:

$$P(A \mid B)P(B) + P(A \mid \overline{B})P(\overline{B})$$

Theorem of Total Probability

- This relation can be generalized with respect to the event space S' ={B_1,B_2,…,B_n) where B_1,B_2,…B_n are collectively exhaustive and mutually exclusive:

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i)$$



The Theorem of Total Probability

- The product of all probabilities from the root of the tree to any node equals the probability of the event represented by that node. P(A) can be computed by summing probabilities associated with all the leaf nodes of the tree.

Bayes' Formula

- The situation often arises in which event A has occurred, but it is not known which of the events $B_1, B_2, \ldots B_n$ has occurred.
- To evaluate $P(B_j|A)$ (the conditional probability that one of the events B_j occurs given that A occurs), by the definition of conditional probability and the theorem of total probability:

$$P(B_j \mid A) = \frac{P(B_j \cap A)}{P(A)} = \frac{P(A \mid B_j)P(B_j)}{\sum_i P(A \mid B_i)P(B_i)}$$

- Bayes' Formula is useful in many applications and forms the basis of a statistical method called **bayesian procedure**.

Blue Waters Example

- Measurements at NCSA's Blue Waters Supercomputer at the University of Illinois indicated that the source of incoming jobs is
  - 15% from Industry
  - 35% from UIUC, and
  - 50% from the Great Lakes Consortium.

- Suppose that some jobs initiated from each of these sites requires a system configuration change (a set-up time). The set-up probabilities are 0.01, 0.05, and 0.02 respectively.

- Find the probability that a job chosen at random at NCSA's Blue Waters system is a ***set-up job***. Also find the probability that a randomly chosen job comes from UIUC, given that it is a *set-up job*.

Blue Waters Example (cont.)

- Define events B_i="Job is from site i" (i=1, 2, 3 for Industry, UIUC, Great Lakes Consortium, respectively) and A="Job requires set-up." Then by the theorem of total probability:

$$P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + P(A \mid B_3)P(B_3)$$
$$= (0.01)\cdot(0.15) + (0.05)\cdot(0.35) + (0.02)\cdot(0.5) = 0.029$$

- Now the second event of interest is $[B_2 \mid A]$, and from Bayes' rule:

$$P(B_2 \mid A) = \frac{P(A \mid B_2)P(B_2)}{P(A)} = \frac{0.05 \cdot 0.35}{0.029} = 0.603$$

- The knowledge that the job is set-up job increases the probability that it came from UIUC from 35 percent to 60 percent

IC Chips Example

- We are given a box containing 5,000 IC chips, of which 1,000 are manufactured by company X and the rest by company Y.
- Ten percent of the chips made by company X and 5 percent of the chips made by company Y are defective.
- If a randomly chosen chip is found to be defective, find the probability that it came from company X.

Given that the chip is defective, it came from company X.

P(A) = chip came from X.

P(B) = chip is defective.

IC Chips Example (cont.)

- Define events A = "Chip is made by company X" and

  B = "Chip is defective."

  P(A) = 1,000/5,000 = 0.2

  (out of a total of 5,000 chips, 300 are defective)

  P(B) = 300/5,000 = 0.06

  Event $A \cap B$ = "Chip is made by company X and is defective"
- Out of 5,000 chips, 100 chips qualify for this statement
- Thus $P(A \cap B)$ = 100/5,000 = 0.02
- Now: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.02}{0.06} = \dfrac{1}{3}$

IC Chips Example (cont.)

- Note: Knowledge of occurrence of Event B has increased the probability of occurrence of Event A. Similarly show that knowledge of occurrence of A has increased the chances of occurrence of B, (P(B|A) = 0.1)

$$\frac{P(A \mid B)}{P(B \mid A)} = \frac{1/3}{0.1} = \frac{0.2}{0.06} = \frac{P(A)}{P(B)}$$

- This property of conditional probabilities holds in general:

$$\frac{P(A \mid B)}{P(B \mid A)} = \frac{P(A \cap B)/P(B)}{P(A \cap B)/P(A)} = \frac{P(A)}{P(B)}$$

- *Repeat this example with A = "Chip is made by company Y"*

Cards Example

- Suppose cards numbered one through ten are placed in a hat, mixed up, and then one of the cards is drawn. If we are told that the number on the drawn card is at least five, then what is the conditional probability that it is ten?

- Let E denote the event that the number of the drawn card is ten, and let F be the event that it is at least five. The desired probability is P(E|F). Now, from the definition we have:

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

- However, EF = E since the number of the card will be both ten and at least five if and only if it is number ten. Hence,

$$P(E \mid F) = \frac{1/10}{6/10} = \frac{1}{6}$$

Example: Family with two children

- A family has two children. What is the conditional probability that both are boys given that at least one of them is a boy? Assume that the sample space S is given by $S = \{(b, b), (b, g), (g, b), (g, g)\}$, and all outcomes are equally likely. ((b, g) means, for instance, that the older child is a boy and the younger child a girl.)

- Letting B denote the event that both children are boys, and A the event that at least one of them is a boy, then the desired probability is given by:

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

$$= \frac{P(\{(b,b)\})}{P(\{(b,b),(b,g),(g,b)\})} = \frac{1/4}{3/4} = \frac{1}{3}$$

Kids party Example

- Suppose that each of three kids at a party throw their hats into the center of the room. The hats are first mixed up and then each kid randomly selects a hat. What is the probability that none of the three kids selects her/his own hat?

- We shall solve this by first calculating the complementary probability that at least one kid selects her/his own hat. Let us denote by E_i , i = 1,2,3, the event that the i^{th} kid selects her/his own hat. To calculate the probability $P(E_1 \cup E_2 \cup E_3)$, we first note that:

$$P(E_i) = \frac{1}{3}, \qquad i = 1,2,3$$

$$P(E_i E_j) = \frac{1}{6}, \quad i \neq j$$

$$P(E_1 E_2 E_3) = \frac{1}{6}$$

Kids party Example (Cont.)

- To see why the previous equations are correct, consider first:

$$P(E_i E_j) = P(E_i)P(E_j \mid E_i)$$

- Now $P(E_i)$, the probability that the i^{th} kid selects her/his own hat, is clearly 1/3 since she/he is equally likely to select any of the three hats.

- On the other hand, given that the i^{th} kid has selected her/his own hat, then there remain two hats that the j^{th} kid may select, and as one of these two is her/his own hat, it follows that with probability 1/2 she/he will select it. That is $P(E_j \mid E_i) = 1/2$, and so:

$$P(E_i E_j) = P(E_i)P(E_j \mid E_i) = \frac{1}{3}.\frac{1}{2} = \frac{1}{6}$$

Kids party Example (Cont.)

- To calculate $P(E_1\ E_2\ E_3)$ we write:

$$P(E_1E_2E_3) = P(E_1E_2)P(E_3 \mid E_1E_2) = \frac{1}{6}P(E_3 \mid E_1E_2)$$

- However, given that the first two kids get their own hats it follows that the third kid must also get her/his own hat (since there are no other hats left). That is, $P(E_3|E_1E_2) = 1$ and so:

$$P(E_1E_2E_3) = \frac{1}{6}$$

- Now, we have that:

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3) &= P(E_1) + P(E_2) + P(E_3) \\ &\quad - P(E_1E_2) - P(E_1E_3) - P(E_2E_3) + P(E_1E_2E_3) \\ &= 1 - \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \end{aligned}$$

- Hence, the probability that none of the kids selects her/his own hat is $1-(2/3) = 1/3$

Minimally-invasive Surgery Example

- Minimally-invasive surgery
  - Small incisions to allow the scope/robotic arms into the body
  - multiple degrees of freedom in instruments to mimic hand/wrist movements

Minimally-invasive Surgery Example

- Suppose that in a given year, in a hospital, a total number of 5000 minimally-invasive surgical procedures are performed.

- Assume that of these procedures, 120 procedures involved device-related adverse events, including:
  - 32 patient injuries,
  - 18 deaths, and
  - 70 system malfunctions.

- Based on the severity of the adverse event, the surgical team may decide to take one of the following recovery actions:
  - Continue the procedure after troubleshooting the problem.
  - Convert the procedure to a non-robotic approach.
  - Reschedule the procedure to a later time.

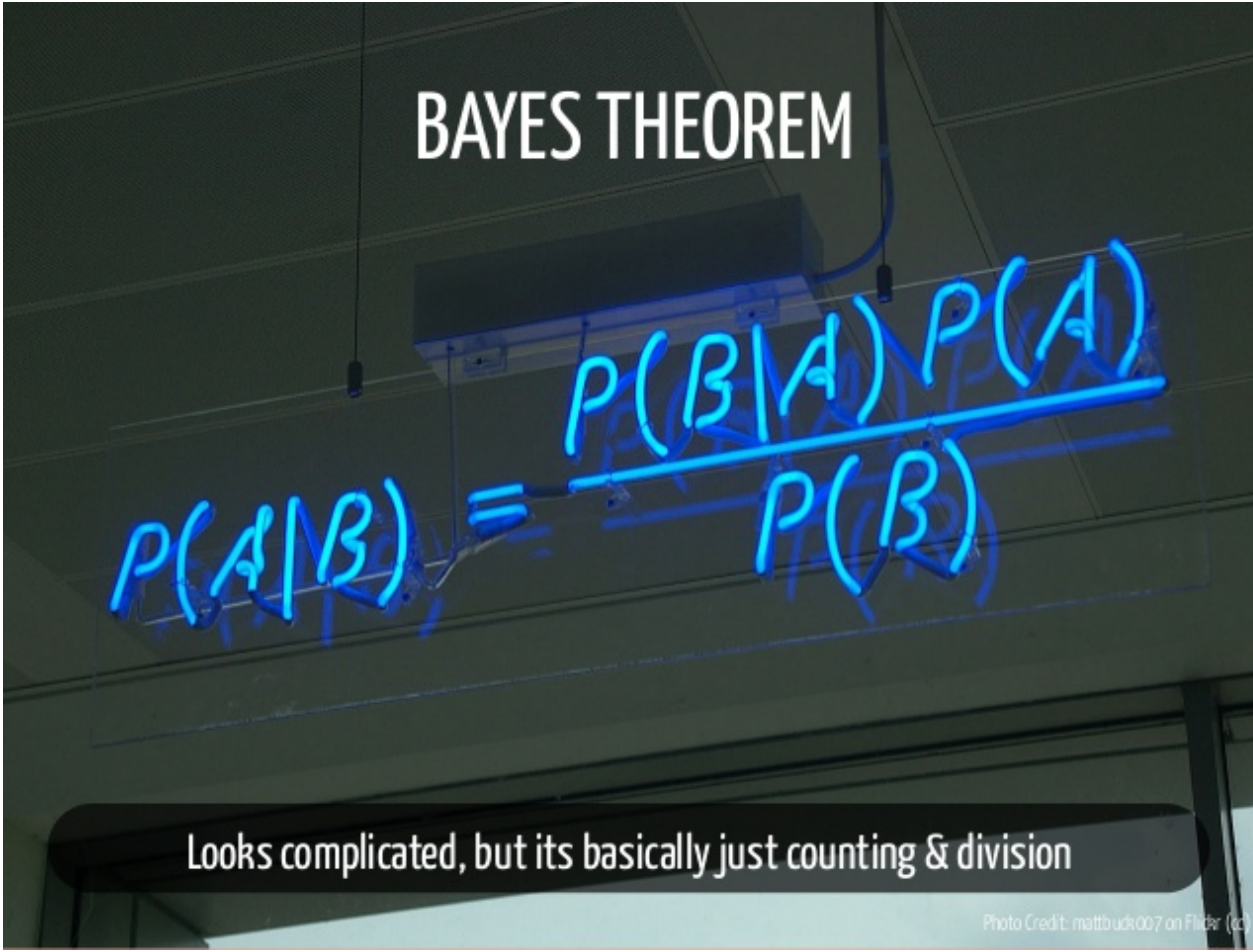
Minimally-invasive Surgery Example

- Assume that 20 of procedures involved injuries and were converted. If an injury happens during a procedure, what would be the probability that it would be converted, *P(C | I)*?

$$P(Conversion \mid Injury) = \frac{P(Conversion \cap Injury)}{P(Injury)} = \frac{20/5000}{120/5000} = \frac{1}{6}$$

- Now assume that we know 2 of procedures involved deaths and were converted, and 36 involved malfunctions and were converted. What is the total probability of a random procedure converted?

$$\begin{aligned} P(Conversion) = {} & P(Conversion \cap Injury) + \\ & P(Conversion \cap Death) + \\ & P(Conversion \cap Malfunction) \\ & = 20/1000 + 2/1000 + 36/1000 = 0.058 \end{aligned}$$


BAYES THEOREM
P(A|B) = P(B|A)P(A) / P(B)
Looks complicated, but its basically just counting & division
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