Probability Theory Basics

ECE 313

Probability with Engineering Applications

Lecture 2

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Today's Topics & Announcements

- **Basic Probability Concepts**
  - Events and Sample Space
  - Algebra of Events
  - Probability Axioms
  - Basic steps for solving problems
  - Combinatorial problems:
    - Permutations with replacement
    - Permutations without replacement
    - Combinations
- **Announcements:**
  - **Homework 1** will be posted on Jan 25th, **due Feb 1st.**
  - **A full lecture schedule including the topics covered and reading for each lecture is on the class website**
  - **First in-class activity on Wednesday Jan 25**
  - **Mini Project 1** will be **posted Feb 1st**
  - Please visit **Piazza** for asking questions and recent updates.

Basic Steps to Solving Problems

- Identify the sample space S
  - The sample space S must be chosen so that all its elements are mutually exclusive and collectively exhaustive, I.e., no two elements can occur simultaneously and one element must occur on any trial.
- Assign probabilities to the elements in S
  - This assumption must be consistent with the axioms A1 through A3
- Identify the events of interests
  - The events are described by statements and need to be recast as subsets of the sample space
- Compute desired probabilities
  - Calculate the probabilities of the events of interest using axioms and any derived laws
- Develop the Insight about the system/experiment

Basic Concepts:
Random Experiment, Sample Space,

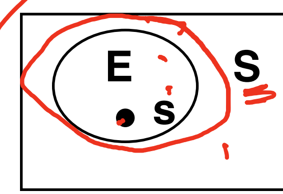
- *Random experiment* is an experiment the outcome of which is not certain
- *Sample Space (S)* is the totality of the possible outcomes of a random experiment
  - in general if a system has n component there are 2^n possible outcomes, each of which can be regarded as a point in an n-dimensional sample space
- *Discrete (countable) sample space* is a sample space which is either
  - *finite*, i.e., the set of all possible outcomes of the experiment is finite or
  - *countably infinite, i.e.,* the set of all outcomes can be put into a one-to-one correspondence with the natural numbers
- *Continuous sample space* is a sample space for which all elements constitute a continuum, such as all the points on a line, all the points in a plane

Events

- An *event* is a collection of certain sample points, i.e., a subset of the sample space
- An *event* is defined as a *statement* whose truth or falsity is determined after the experiment
- The set of all outcomes for which the statement is true defines the subset of the sample space corresponding to the event

Events (cont.)

- *Elementary event* is the event *{s}* consisting of a single sample point
- Bringing the Definitions together
  - E is an event defined in the sample space S; E is a subset of S
  - Outcome of a specific "trial" is an element {s}
  - "s" is an element in E ==> "event E has occurred"
  - Note "s" may be an element in multiple events (i.e., only one outcome of the experiment but many events may occur)



- *Universal event* is the entire sample space S
- *The null set* ∅ is a null or impossible event

Algebra of Events

- Consider an example of a computer system with five identical processors.
- Let a random experiment consists of checking the system to see how many CPUs are currently available.
- A CPU is in one of two states : busy (0) and available (1).
- The sample space S has 2^5 = 32 sample points

Algebra of Events (cont.)

$s_0 = (0, 0, 0, 0, 0)$	$s_{16} = (1, 0, 0, 0, 0)$
$s_1 = (0, 0, 0, 0, 1)$	$s_{17} = (1, 0, 0, 0, 1)$
$s_2 = (0, 0, 0, 1, 0)$	$s_{18} = (1, 0, 0, 1, 0)$
$s_3 = (0, 0, 0, 1, 1)$	$s_{19} = (1, 0, 0, 1, 1)$
$s_4 = (0, 0, 1, 0, 0)$	$s_{20} = (1, 0, 1, 0, 0)$
$s_5 = (0, 0, 1, 0, 1)$	$s_{21} = (1, 0, 1, 0, 1)$
$s_6 = (0, 0, 1, 1, 0)$	$s_{22} = (1, 0, 1, 1, 0)$
$s_7 = (0, 0, 1, 1, 1)$	$s_{23} = (1, 0, 1, 1, 1)$
$s_8 = (0, 1, 0, 0, 0)$	$s_{24} = (1, 1, 0, 0, 0)$
$s_9 = (0, 1, 0, 0, 1)$	$s_{25} = (1, 1, 0, 0, 1)$
$s_{10} = (0, 1, 0, 1, 0)$	$s_{26} = (1, 1, 0, 1, 0)$
$s_{11} = (0, 1, 0, 1, 1)$	$s_{27} = (1, 1, 0, 1, 1)$
$s_{12} = (0, 1, 1, 0, 0)$	$s_{28} = (1, 1, 1, 0, 0)$
$s_{13} = (0, 1, 1, 0, 1)$	$s_{29} = (1, 1, 1, 0, 1)$
$s_{14} = (0, 1, 1, 1, 0)$	$s_{30} = (1, 1, 1, 1, 0)$
$s_{15} = (0, 1, 1, 1, 1)$	$s_{31} = (1, 1, 1, 1, 1)$

Algebra of Events (cont.)

- Let the events E_1 and E_2 be defined as follows:

E_1 - "At least four CPUs are available" - is given by:

$E_1 = \{(0, 1, 1, 1, 1), (1, 0, 1, 1, 1), (1, 1, 0, 1, 1), (1, 1, 1, 0, 1),$
$(1, 1, 1, 1, 0), (1, 1, 1, 1, 1)\} = \{s_{15}, s_{23}, s_{27}, s_{29}, s_{30}, s_{31}\}$

$\overline{E}_1$ (complement) = S - E_1 = {all points not in E_1}

$\overline{E}_1 = \{s_o$ thro s_{14}, s_{16} thro s_{22}, s_{24} thro s_{26}, $s_{28}\}$

E_2 - "CPU 1 is available" - is given by:

$E_2 = \{s_{16}$ thro $s_{31}\}$

$s_0 = (0, 0, 0, 0, 0)$	$s_{16} = (1, 0, 0, 0, 0)$
$s_1 = (0, 0, 0, 0, 1)$	$s_{17} = (1, 0, 0, 0, 1)$
$s_2 = (0, 0, 0, 1, 0)$	$s_{18} = (1, 0, 0, 1, 0)$
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$s_{15} = (0, 1, 1, 1, 1)$	$s_{31} = (1, 1, 1, 1, 1)$

Algebra of Events (cont.)

- The *intersection* E_3 of E_1 and E_2 is given by:

$$E_3 = E_1 \cap E_2 = \{s \in S \mid s \text{ is an element of both } E_1 \text{ and } E_2\}$$

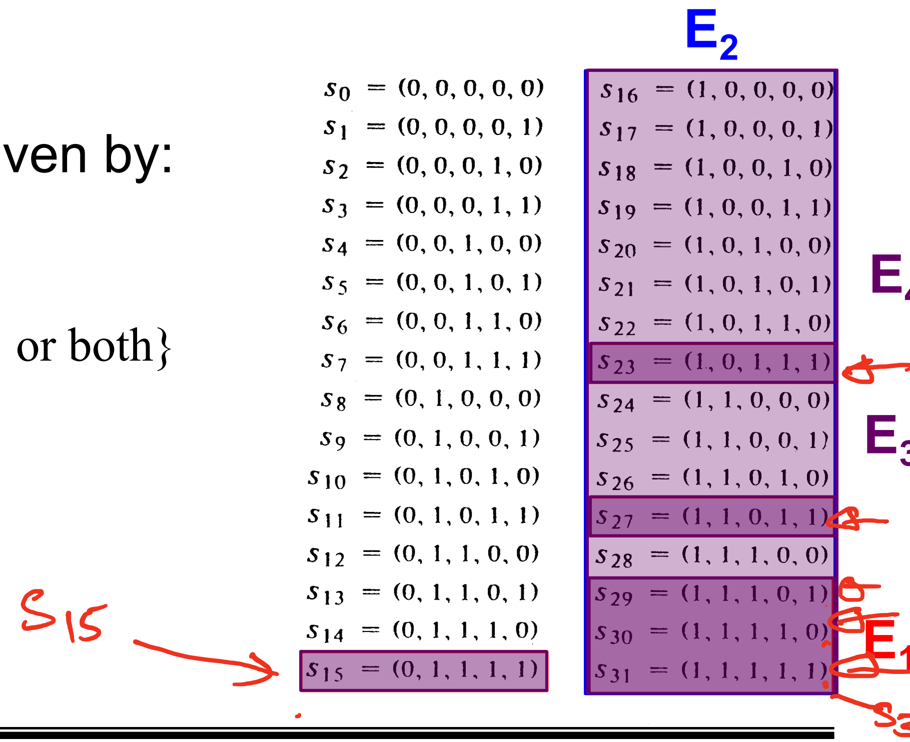
$$= \{s \in S \mid s \in E_1 \text{ and } s \in E_2\} = \{s_{23}, s_{27}, s_{29}, s_{30}, s_{31}\}$$

- The *union* E_4 of E_1 and E_2 is given by:

$$E_4 = E_1 \cup E_2$$

$$= \{s \in S \mid \text{either } s \in E_1 \text{ or } s \in E_2 \text{ or both}\}$$

$$= \{s_{15} \text{ through } s_{31}\}$$



Algebra of Events (cont.)

- In general: $|E_4| = |E_1 \cup E_2| \leq |E_1| + |E_2|$

  where $|A|$ = the number of elements in the set (Cardinality)

- *Mutually exclusive or disjoint events* are two events for which

  $$A \cap B = \varnothing$$

- Definition of *union* and *intersection* extend to any finite number of sets

  s is an element in E_1 or E_2…….

  $$\bigcup_{i=1}^{n} E_i = E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n$$

  s is an element in E_1 & E_2 &…….

  $$\bigcap_{i=1}^{n} E_i = E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n$$

Algebra of Events
Laws or Axioms

- A, B, C are arbitrary sets (or events), S is the universal set or event
- (E1) Commutative laws:

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A$$

- (E2) Associative laws:

$$A \cup (B \cup C) = (A \cup B) \cup C, \qquad A \cap (B \cap C) = (A \cap B) \cap C$$

- (E3) Distributive laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- (E4) Identity laws:

$$A \cup \varnothing = A, \qquad A \cap S = A$$

- (E5) Complementation laws:

$$A \cup \overline{A} = S, \qquad A \cap \overline{A} = \varnothing$$

Sample

Algebra of Events

Useful Relations

- (R1) Idempotent laws:

  $A \cup A = A$, $A \cap A = A$

- (R2) Domination laws:

  $A \cup S = S$, $A \cap \varnothing = \varnothing$

- (R3) Absorption laws:

  $A \cap (A \cup B) = A$, $A \cup (A \cap B) = A$

- (R4) DeMorgan's laws:

  $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

- (R5) $\overline{(\overline{A})} = A$

- (R6) $A \cup (\overline{A} \cap B) = A \cup B$

Algebra of Events (cont.)

- A list of events $A_1, A_2, \ldots, A_n$ is said to be
  - composed of *mutually exclusive events* iff:

$$A_i \cap A_j = \begin{cases} A_i & \text{if } i = j \\ \varnothing & \text{otherwise} \end{cases}$$

(intuitively: a list has mutually exclusive events if no point in the sample space is included in more than one event in the list)

  - *collectively exhaustive* iff:

$$A_1 \cup A_2 \cup \ldots \cup A_n = S$$

(each point in the sample space is included in at least one event in the list)

Probability Axioms

- Let *S* be a sample space of a random experiment and *P(A)* be the probability of the event *A*
- The probability function *P(.)* must satisfy the three following axioms:
- **(A1)** For any event A, P(A) ≥ 0

  (probabilities are nonnegative real numbers)
- **(A2)** P(S) = 1

  (probability of a certain event, an event that must happen is equal 1)
- **(A3)** $P(A \cup B) = P(A) + P(B)$, whenever A and B are mutually exclusive events, i.e., $A \cap B = \varnothing$ ← intersection is a null set.

  (probability function must be additive)

Probability Axioms (cont.)

To deal with infinite sample space the axiom (A3) needs to be modified:

- **(A3')** For any countable sequence of events $A_1, A_2, \ldots, A_n \ldots$, that are mutually exclusive (that is $A_j \cap A_k = \varnothing$ whenever $j \neq k$)

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

- The conventional probability theory follows from the three axioms (A1 through A3') of probability measure and the five axioms (E1 through E5) of the algebra of events.

Probability Axioms
Useful Relationships

P(S) = 1

- **(Ra)** For any event A, $P(\overline{A}) = 1 - P(A)$
- **(Rb)** If ∅ is the impossible event, then $P(\varnothing) = 0$
- **(Rc)** If A and B are any events, not necessarily mutually exclusive, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **(Rd)**(generalization of Rc) If $A_1, A_2, \ldots, A_n$ are any events, then

$$P(\bigcup_{i=1}^{n} A_i) = P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_i P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j)$$

$$+ \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) + \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

where the successive sums are over all possible events, pairs of events, triples of events, and so on.

(Can prove this relation by induction (see class web site))

Probability Axioms (cont.)

To avoid mathematical difficulties following definitions are introduced

- *"Class of events"* (*F*) defines a particular class of subsets of *S* that is measurable
- *F* is closed under countable unions as well as under complementation and called a σ-field of subsets of *S*
- Now *probability space* or *probability system* is defined as a triple (S, *F,* P) where:
  - S is a sample space
  - *F* is a σ-field of subsets of *S* which includes *S*
  - P is probability measure of *F*
- P is a function with domain *F* and range [0, 1], which satisfies axioms A1, A2 and A3'
- P assigns a number between [0, 1] to any event in *F*
- In general, *F* does not include all possible subsets of S and the subsets (events) included in *F* are called measurable

What Have We Gained by Defining the Axioms?

- The axioms ensure the same rules apply no matter how we assign probabilities, initially.
- Does not require that all subsets of S be "events"
  - Axiom 1 says: For all "events", A $0 \leq P(A) \leq 1$

    if A is not an event, then we don't assign a probability to it.
- Restrictions
  - S is always an event
  - If A is an event, $\bar{A}$ is also an event.
  - Subsets $A_1, A_2 \ldots. A_n$ are events, their union and intersections are also events.
  - A collection of events which satisfies the 3 axioms is called F or σ Field.
  - σ Field contains S and is closed under complementations, countable unions, and intersections.

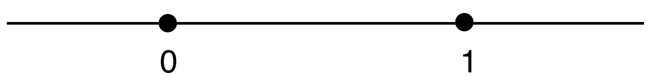
Elements of Sample Space

- The outcomes of an experiment
- Each element is a point in the sample space (of one or more dimensions)
- Examples

1-dimensional sample space:

A single component with two states (working is represented by 1; failed is represented by 0)

(tossing a fair coin)



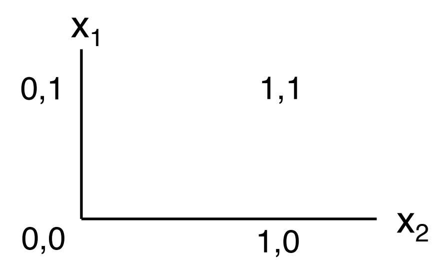
Represented by one variable x=(0,1)

2-dimensional sample space:

A system of two components x_1,x_2

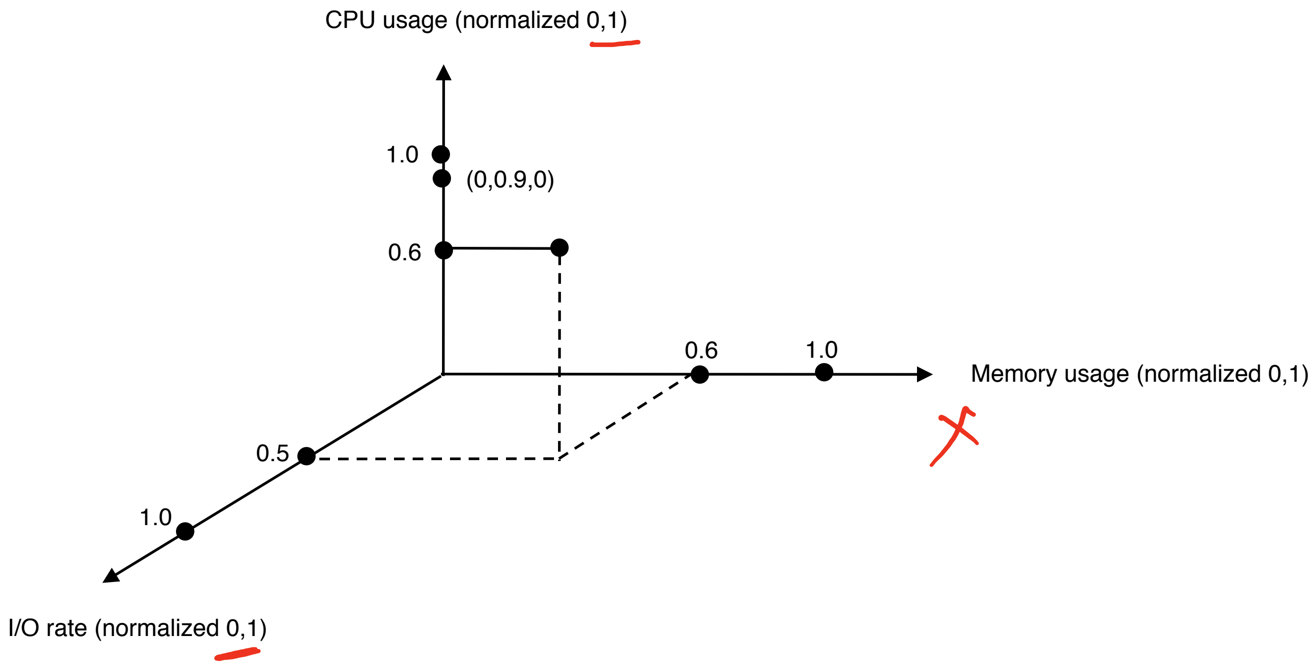
The status of the system (x_1,x_2):

(0,0) (0,1) (1,0) (1,1)



Elements of Sample Space (cont.)

3-dimensional sample space (representing the workload on a system):



Overview

- Summarizing:
  - A1 For any event A $P(A) \geq 0$
  - A2 $P(S) = 1$ (Prob of a certain event)
  - A3 $P(A \cup B) = P(A) + P(B)$

    A & B are mutually exclusive

    $A \cap B = \emptyset$

- Next:
  - Basic steps in problem solving
  - Combinatorial methods

Basic Steps to Solving Problems

- Identify the sample space S
  - The sample space S must be chosen so that all its elements are mutually exclusive and collectively exhaustive, I.e., no two elements can occur simultaneously and one element must occur on any trial.
- Assign probabilities to the elements in S
  - This assumption must be consistent with the axioms A1 through A3
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Examples

- Consider a wireless cell with five channels

(compare 5 CPUs) previosly

- ***Step 1.*** A sample space consists of 32 points, each represented by a 5-tuple of 0's and 1's (0 = busy; 1 = available)

- ***Step 2.*** We assume that each sample point is equally likely and consequently we assign a probability of 1/32 to each point

Examples (Cont.)

- ***Step 3.*** Assume that we need to determine the probability that a call is not blocked, given that a conference call needs at least three channels for its execution. The event E of interests, then, is "Three or more channels are available"
- $E = \{s_7, s_{11}, s_{13}, s_{14}, s_{15}, s_{19}, s_{21}, s_{22}, s_{23}, s_{25}, s_{26}, s_{27}, s_{28}, s_{29}, s_{30}, s_{31}\}$.

  16 possibilities each prob of 1/32
- ***Step 4.*** Event E can be expressed as a union of mutually exclusive events. The probability of these elementary events is 1/32 thus:

$$P(E) = \sum_{s_i \in E} P(s_i) = \frac{1}{2}$$

successful conf call.

→ 50% of making the call

Examples (Cont.)

- A computer's physical memory space can be divided into several fixed-sized contiguous blocks, called memory pages. Some x86 processors can support pages of different sizes. Assume that an Intel 64-bit processor supports three page types of sizes 4KB, 4MB, and 1GB. Two pages are selected from the processor memory at random and are examined to see if they are of different sizes or not.

Answer:

- An event in this experiment consists of a draw which has a pair of memory pages, (e.g. {4K,1G}). Total number of such events is $(3 \times 3) = 9$. (The first draw can be performed in three different ways and so is the second draw – sampling with replacement) Then the sample space would be 9.

Combinatorial Problems

- We are often concerned with selecting some number of objects from a total
  - Defects
  - Allocating processors for scheduling
  - Performance measurement
- Sample Space consisting of a finite number (n) of points (elements, sample points, and outcomes

$$P(s_i) = p_i$$

$$\sum_{i=1}^{n} p_i = 1$$

- For any event E, If we assume that all s_i are equally likely: $P(s_i) = p_i = = 1/n$

  P(E) = # pts in E / # pts in S

Combinatorial Problems

- In the special case when S = {s1, …, sn} and P(si) = pi = 1/n
- If the event *E* consists of *k* sample points, then

$$P(E) = \frac{\text{number of points in E}}{\text{number of points in S}} = \frac{\text{favorable outcomes}}{\text{total outcomes}} = \frac{k}{n}$$

Example 1

- Consider the following if-statement in a program:

  if B then s_1 else s_2

- In random experimental "observing" of two successive executions of the if-statement, the sample space is:

$$S = \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\}$$
$$= \{t_1, t_2, t_3, t_4\}$$

- On the basis of experimental evidence:

$$P(t_1) = 0.34, P(t_2) = 0.26, P(t_3) = 0.26, P(t_4) = 0.14$$

← Actual data

- The events of interest are:

  E_1="At least one execution of the statement s_1."

  E_2="Statement s_2 is executed the first time."

- It is easy to see that:

$$E_1 = \{t_1, t_2, t_3\}$$
$$E_2 = (t_3, t_4\}$$
$$P(E_1) = P(t_1) + P(t_2) + P(t_3) = 0.86$$
$$P(E_2) = P(t_3) + P(t_4) = 0.4$$

Example 2

- A group of four integrated-circuit (IC) chips consists of two good chips, labeled g_1 and g_2, and two defective chips, labeled d_1 and d_2. If three chips are selected at random from this group, what is the probability of the event:

E="Two of the three selected chips are defective."

$S=[g_1,g_2,d_1][g_1,g_2,d_2][g_1,d_1,d_2][g_2,d_1,d_2]$
under the equi-probable assumption.

$P(E) = 2/4= 1/2$

Permutations with Replacement

- Ordered samples of size k, with replacement *(permutations with replacement) P(n, k)*

  gives the number of ways we can select k objects among n objects where order is important and when the same object is allowed to be repeated any number of times; the required number is n^k

- Example: Find the probability that some randomly chosen k-digit decimal number is a valid k-digit octal number.

  The sample space is $S = \{(x_1, x_2, \ldots x_k) \mid x_1, \in \{0, 1, 2, \ldots, 9\}\}$

  The events of interests is $E = \{(x_1, x_2, \ldots x_k) \mid x_1, \in \{0, 1, 2, \ldots, 7\}\}$

  $|S| = 10^k$ and $|E| = 8^k$ ----> $P(E) = |E| / |S| = 8^k / 10^k$

Permutations without Replacement

- Ordered Samples of size k, without replacement *(permutations without replacement)*
- Counts the number of ordered sequences without repetition of the same element(s); the number is given by:

$$n(n-1)\ldots.(n-k+1) = \frac{n!}{(n-k)!} \quad k = 1, 2, \ldots, n$$

- Example: Find the probability that a randomly chosen three-letter sequence will not have any repeat letters.

  Let $I = \{a, b, \ldots, z\}$ be the alphabet of 26 letters

  $S = \{(\alpha, \beta, \gamma) \mid \alpha \in I, \beta \in I, \gamma \in I\}$

  $E = \{(\alpha, \beta, \gamma) \mid \alpha \in I, \beta \in I, \gamma \in I, \alpha \neq \beta, \beta \neq \gamma, \alpha \neq \gamma\}$

  $|E| = P(26, 3) = 15{,}600$; $|S| = 26^3 = 17{,}576$

  $P(E) = 15{,}600 / 17{,}576 = 0.8876$

Combinations

- Unordered sample of size k, without replacement *(combinations)* gives the number of unordered sets of distinct elements; the number is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Example: If a box contains 75 good IC chips and 25 defective chips, and 12 chips are selected at random, find the probability that at least one chip is defective.
- The event of interest is E = "At least one chip is defective"; we use a complementary event E' = "No chip is defective"

$|E'| = \binom{75}{12}$ $|S| = \binom{100}{12}$

P(E') = |E'| / |S| = (75! * 88!) / (63! * 100!) and P(E) = 1 - P(E')

no defects

Binomial Theorem

- The values $\binom{n}{k}$ are called *binomial coefficients*

Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Combinatorial Proof of the Binomial Theorem:

- Consider the product $(x1 + y1)(x2 + y2)\cdots(xn + yn)$
- Its expansion consists of the sum of 2*n* terms, each term being the product of *n* factors. Furthermore, each of the 2*n* terms in the sum will contain as a factor either *xi* or *yi* for each *i* = 1,2,...,*n*.
- Now, how many of the 2*n* terms in the sum will have *k* of the *xi*'s and (*n*−*k*) of the *yi*'s as factors?
- As each term consisting of *k* of the *x* 's and (*n* − *k*) of the *y* 's corresponds to a choice of a group of *k* from the *n* values *x*1,*x*2,...,*xn*, there are $\binom{n}{k}$ such terms.

Example 3: Selecting your favorite Apps

- Suppose we have 100 different Apps in the AppStore. In each of the following cases, what are the number of possible outcomes?

a. We ask 3 different students about their favorite App.

b. We ask one student to give a ranking of the best 3 Apps that the student has used.

c. We ask one student to pick 3 of her/his favorite App to recommend

Example 3 - Solution

a. Permutations with replacement:
   – $(100)^3 = 1{,}000{,}000$

b. Permutations without replacement:
   – $100 \times 99 \times 98 = 100! / (100-3)! = 100! / 97! = 970{,}200$

c. Combinations:
   – $\binom{100}{3} = 100! / (3!\,97!) = 161{,}700$

Example 4: Picking the "right" Shoe(s)

- A bag contains n pairs of shoes in distinct styles and sizes. You pick two shoes at random from the bag. Note that this is sampling without replacement.
  - a) What is the probability that you get a pair of matching shoes?
  - b) What is the probability of getting one left shoe and one right shoe?

- Suppose now that $n \geq 2$ and that you choose 3 shoes at random from the bag.
  - c) What is the probability that you have a pair of matching shoes among the three that you have picked?
  - d) What is the probability that you picked at least one left shoe and at least one right shoe?

Example 2 - Solution

a. Consider pairs of shoes as being unordered, that is, it does not matter which shoe of the pair is picked first. There are $\binom{2n}{2}$ pairs in total and among these n are matching thus the probability is:

$$\frac{n}{\binom{2n}{2}} = \frac{n}{2n(2n-1)/2} = \frac{1}{2n-1}$$

b. To get one left shoe and one right shoe, we need the second shoe to be of the opposite type of the first shoe. After picking the first shoe, among the 2n − 1 remaining shoes, n are of the opposite type. Thus, the probability of getting one left and one right shoe is n/(2n − 1).

Example 2 - Solution

c. There are *n(2n − 2)* unordered triples including a matching pair, among the $\binom{2n}{3}$ possible triples. Thus, the answer is:

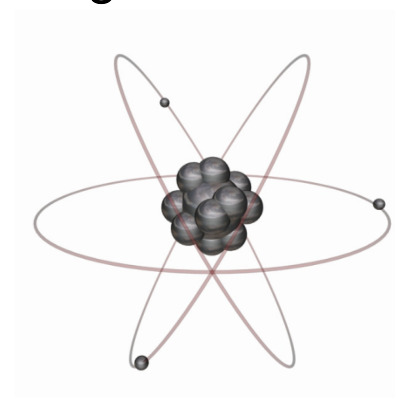
$$\frac{n(2n-2)}{\binom{2n}{3}} = \frac{n(2n-2)}{2n(2n-1)(2n-2)/6} = \frac{3}{2n-1}$$

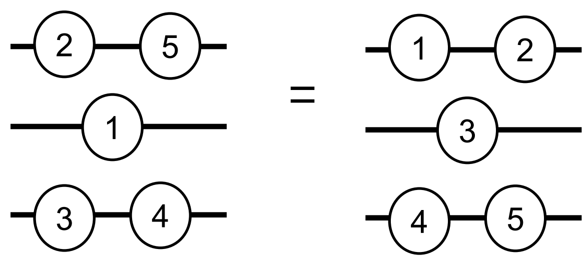
d. There are $2.\binom{n}{3}$ all-left or all-right triples. Thus the probability of a mixed triple is:

$$1 - \frac{2\binom{n}{3}}{\binom{2n}{3}} = 1 - \frac{n(n-1)(n-2)/3}{2n(2n-1)(2n-2)/6} = 1 - \frac{(n-2)}{2(2n-1)} = \frac{3n}{4n-2}$$

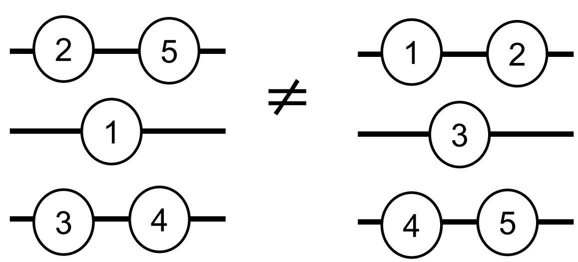
Example 3

- Assume that you have five electrons and three orbits. In how many ways can you distribute the electrons into orbits so that no orbit contains more than two electrons? (The ordering of electrons on each orbit is not important).
  - Assuming that the electrons are identical:





  - Assuming that the electrons are distinguishable (not identical):



Example 3 - Solution

- Assuming that the electrons are identical:
  - There are **3 different ways** to assign 5 identical electrons to 3 different orbits. We can show it as a sequence of number of electrons assigned to each orbit:

    (Orbit 1, Orbit 2, Orbit 3) = (1, 2, 2) or (2, 1, 2) or (2, 2, 1)

- Assuming that the electrons are distinguishable (not identical):
  - If the electrons are distinguishable, there are 3 different ways to assign the 5 electrons to 3 orbits and in each case we have 30 ways to select the electrons, so in total there are **90 different ways**:
    - For case (1,2,2): $\binom{5}{1}\binom{4}{2}\binom{2}{2} = 30$
    - For case (2,1,2): $\binom{5}{2}\binom{3}{1}\binom{2}{2} = 30$
    - For case (2,2,1): $\binom{5}{2}\binom{3}{2}\binom{1}{1} = 30$

Example 4

- The following for loop is executed in a program. *p* is a variable that can be either TRUE or FALSE each time through the loop, A and B are statements that can be executed. In how many ways, exactly 3 ‘A’s can occur?

```
for (int i = 0; i < 32; i++)
{
        if (p == TRUE)
                A;
        else
                B;
}
```

- Answer: $\binom{32}{3}$