University of Illinois Spring 2016

ECE 313: Final Exam

Monday, May 9, 2016 7 p.m. — 10 p.m.
ECEB 1002 - ECEB 1013

1. [**14 points**] Consider a five-sided die, with equiprobable sides labeled $1, 2, 3, 4, 5$.

   (a) If you roll the die twenty times, what is the probability that for exactly five of the rolls, the outcome is in $\{2, 3\}$?

   Solution: In each roll, the event $A = \{\text{either 2 or 3 shows}\}$ has probability $\frac{2}{5}$. The rolls are mutually independent, so the number of times event A occurs in twenty rolls has the $Binomial(20, 2/5)$ distribution. Therefore, the probability A happens five times in 20 rolls is $\binom{20}{5}\left(\frac{2}{5}\right)^5\left(\frac{3}{5}\right)^{15}$.

   (b) If you roll the die twenty times, given that the sixth roll is a 4, what is the probability that for exactly five of the twenty rolls, the outcome is in $\{2, 3\}$?

   Solution: Each roll is independent of other rolls, hence the number of times event A occurs in twenty rolls given that it did not occur on the sixth roll is $Binomial(19, 2/5)$. Therefore,
   $P\{A \text{ happens five times in twenty rolls}|\text{sixth roll is a } 4\} = \binom{19}{5}\left(\frac{2}{5}\right)^5\left(\frac{3}{5}\right)^{14}$.

   (c) If you know that the sixth roll is a 2, what is the probability that the next time that either a 2 or 3 shows is the fifteenth roll?

   Solution: Each roll is independent of other rolls, hence the number of rolls until event A occurs after the sixth roll is $Geometric(2/5)$. Therefore, $P\{\text{it takes nine more rolls after the sixth roll for either 2 or 3 to show}|\text{sixth roll is a } 2\} = \left(\frac{3}{5}\right)^8\left(\frac{2}{5}\right)$.

2. [**14 points**] A random variable X is drawn from one of two possible distributions:
   H_1 : Poisson with parameter $\lambda = 1$ or H_0 : Poisson with parameter $\lambda = e$.
   You may need the fact $e \approx 2.7$.

   (a) If $X = 3$, which hypothesis does the maximum likelihood (ML) rule choose?

   Solution: The pmf of X under H_1 is given by $p_1(k) = \frac{e^{-1}}{k!}$ and the pmf of X under H_0 is given by $p_0(k) = \frac{e^{-e+k}}{k!}$. Thus, the likelihood ratio is $\Lambda(k) = \frac{p_1(k)}{p_0(k)} = e^{e-1-k}$. Since $\Lambda(3) = e^{e-4} < 1$, the ML decision for observation $X = 3$ is H_0.

   (b) If $X = 3$, what hypothesis does the MAP-rule choose if $\pi_1 = e^2\pi_0$?

   Solution: So $\frac{\pi_0}{\pi_1} = e^{-2}$. Since $\Lambda(3) = e^{e-4} > e^{-2}$, the MAP decision for observation $X = 3$ is H_1.

3. [**14 points**] Suppose $S = X_1 + X_2 + X_3$ where X_1, X_2, X_3 are mutually independent and X_i has the Bernoulli distribution with parameter $p_i = \frac{i}{5}$ for $i \in \{1, 2, 3\}$.

   (a) Find $P(S = 1)$.

Solution:

$$\begin{aligned} P(S=1) &= P(X_1=1, X_2=0, X_3=0) + P(X_1=0, X_2=1, X_3=0) \\ &\quad + P(X_1=0, X_2=0, X_3=1) \\ &= 0.2(1-0.4)(1-0.6) + (1-0.2)0.4(1-0.6) + (1-0.2)(1-0.4)0.6 \\ &= 0.048 + 0.128 + 0.288 = 0.464 \end{aligned}$$

(b) Find $P(X_1 = 1|S = 1)$.
Solution:

$$\begin{aligned} P(X=1|S=1) &= \frac{P(X=1, S=1)}{P(S=1)} \\ &= \frac{P(X_1=1, X_2=0, X_3=0)}{P(S=1)} = \frac{0.048}{0.464} = \frac{6}{58}. \end{aligned}$$

4. [**14 points**] The NASA Long Duration Exposure Facility (LDEF) encountered an average of 10 impacts per day over a 5.7 year period. Suppose the timing of the impacts is well modeled by a Poisson process with rate parameter 10/day.

   (a) What is the probability of exactly 100 impacts in a one week period?
   Solution: The number of impacts in a week (seven days) has the Poisson distribution with mean 70. So the requested probability is $\frac{e^{-70}(70)^{100}}{100!}$.

   (b) Given there are 60 impacts in the first three days, what is the conditional mean number of impacts in the first day?
   Solution: Given 60 counts in the three day time period, the arrival times of the counts are uniformly distributed over the time period, so each would occur in the first day with probability 1/3. So the conditional mean for one day is 60/3=20.

   (c) Given there are 60 impacts in the first three days, what is the conditional mean number of impacts in the first *week*?
   Solution: There are 60 impacts in the first three days, and ten impacts are expected for each of the remaining four days of the week. This gives a total of $60+4\cdot10$, or 100, impacts for the week, given there were 60 impacts the first three days.

5. [**15 points**] Two random variables X and Y take values u and v, respectively, in the set $\{0, 1, 2, 3\}$.

   (a) The table below partially gives the joint pmf and the marginal pmfs. It is known that: $P\{X = 3, Y \geq 2\} = 0.05$ and $P\{X = Y\} = 0.06$. Complete the table by filling in the seven underlined blanks.

		Y				
		$v=0$	$v=1$	$v=2$	$v=3$	$p_X(u)$
	$u=0$	0.01	___	0.1	___	0.3
X	$u=1$	0	0	___	0.1	0.2
	$u=2$	0.09	0.01	0.05	0.05	0.2
	$u=3$	0.05	0.2	___	___	___
	$p_Y(v)$	0.15	___	0.3	0.25	

Solution: $P\{X = Y\}$ is the sum of the diagonal entries of the joint pmf, so $p(3,3) = 0$. Then the information $P\{X = 3, Y \geq 2\} = 0.05$ yields $p(3,2) = 0.05$. Fill in the remainder of the blanks using the fact the marginal pmfs are given by the row or column sums of the joint pmf.

	$v=0$	$v=1$	$v=2$	$v=3$	$p_X(u)$
$u=0$	0.01	$\underline{0.09}$	0.1	$\underline{0.1}$	0.3
$u=1$	0.0	0.0	$\underline{0.1}$	0.1	0.2
$u=2$	0.09	0.01	0.05	0.05	0.2
$u=3$	0.05	0.2	$\underline{0.05}$	$\underline{0.0}$	$\underline{0.3}$
$p_Y(v)$	0.15	$\underline{0.3}$	0.3	0.25	

(b) Given $Y = 0$, what is the conditional probability $X = 3$?
Solution: This part asks for $P(X = 3|Y = 2) = \frac{0.05}{0.15} = 1/3$.

(c) The joint pmf of W and Z and its marginals are shown below. Determine the conditional probability: $P\{Z \text{ is even}|W \text{ is odd}\}$.

			Z			
		$v=0$	$v=1$	$v=2$	$v=3$	$p_W(u)$
	$u=0$	0.01	0.09	0.1	0.05	0.25
W	$u=1$	0.0	0.05	0.1	0.05	0.2
	$u=2$	0.09	0.01	0.00	0.05	0.15
	$u=3$	0.1	0.2	0.05	0.05	0.4
	$p_Z(v)$	0.2	0.35	0.25	0.20	

Solution: This is $P\{Z = 0,2|W = 1,3\} = \frac{P\{Z=0,2,W=1,3\}}{P\{W=1,3\}} = \frac{0.0+0.1+0.1+0.05}{0.2+0.4} = 5/12$.

6. **[18 points]** Let X and Y be random variables with joint pdf $f_{X,Y}(u,v) = \frac{1}{2}e^{-u}$ for $u \geq 0, v \in (1,2) \cup (3,4)$, and zero else.

(a) Are X and Y independent? Indicate why or why not.
Solution: If $X \sim Exp(1)$ and $Y \sim Uniform\left((1,2) \cup (3,4)\right)$, then $f_{X,Y}(u,v) = f_X(u)f_Y(v)$ everywhere on the plane. Hence they are independent.

(b) Obtain the marginal $f_X(u)$ for all u.
Solution: $X \sim Exp(1)$, therefore $f_X(u) = e^{-u}$ for $u \geq 0$ and zero else.

(c) Obtain the conditional pdf $f_{Y|X}(v|u)$ for all u, v.
Solution: From part (a), we know that X and Y are independent, hence $Y \sim Uniform\left((1,2) \cup (3,4)\right)$ for all $u \geq 0$ and it is undefined else; that is, if $u \geq 0$, $f_{Y|X}(v|u) = \frac{1}{2}$ for $v \in \{(1,2) \cup (3,4)\}$ and zero else.

(d) Let $Z = X + Y$. Obtain $P\{Z \leq 3\}$.
Solution:

$$P\{Z \leq 3\} = \int_1^2 \int_0^{3-v} \frac{1}{2}e^{-u}dudv = \frac{1}{2}\int_1^2 \left(1 - e^{-(3-v)}\right) dv = \frac{1}{2}\left(1 - e^{-1} + e^{-2}\right)$$

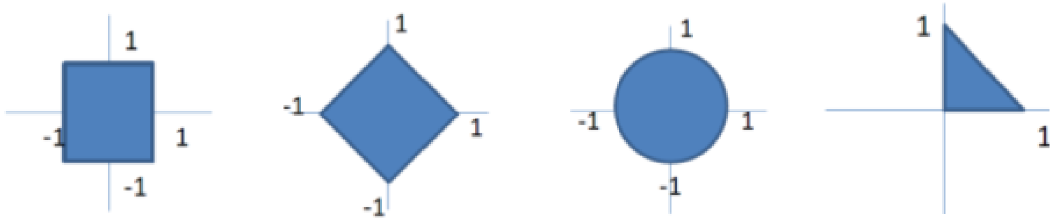


Figure a Figure b Figure c Figure d

7. [**20 points**] Continuous random variables X, Y are uniformly distributed over regions $\mathcal{C}_a$,$\mathcal{C}_b$,$\mathcal{C}_c$ and $\mathcal{C}_d$ which are shown in Figures (a),(b),(c) and (d), respectively. In each of the cases find the means μ_X, μ_Y and the covariance, $\mathrm{Cov}(X, Y)$. Numerical answers are expected.

   Solution: In each of Figures (a), (b), and (c), the regions are symmetric around the origin for both X and Y dimensions. Hence $\mu_X = \mu_Y = 0$. For these regions we also have $E[XY] = 0$, again due to the symmetry around the origin. Thus $\mathrm{Cov}(X, Y) = 0$. For Figure (d), the scenario requires explicit computations. The pdf over the triangular region has value 2, so

   $$\mu_X = 2\int_0^1 \int_0^{1-u} u\, dv\, du = 2\int_0^1 u(1-u)\, du = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}.$$

   Due to symmetry, we have $\mu_X = \mu_Y$. Similarly,

   $$E[XY] = 2\int_0^1 \int_0^{1-u} uv\, dv\, du = 2\int_0^1 \frac{u}{2}(1-u)^2\, du = \frac{1}{12}.$$

   Thus $\mathrm{Cov}(X, Y) = \frac{1}{12} - \frac{1}{9} = -\frac{1}{36}$.

8. [**20 points**] Suppose X and Y have a bivariate Gaussian joint distribution with $E[X] = 1$, $E[Y] = 2$, $\mathrm{Var}(X) = 4$, $\mathrm{Var}(Y) = 16$, and the correlation coefficient $\rho = -0.5$.

   (a) Are $X - Y$ and $X + Y$ independent? Why or why not?
   Solution: Since X and Y are jointly Gaussian, $X - Y$ and $X + Y$ are jointly Gaussian too. Hence they are independent if and only if they are uncorrelated.

   $$\mathrm{Cov}(X - Y, X + Y) = \mathrm{Var}(X) - \mathrm{Var}(Y) = 4 - 16 = -12 \neq 0,$$

   so $X - Y$ and $X + Y$ are *not* independent.

   (b) Find $P\{X - Y \geq 5\}$.
   Solution: $X - Y$ is a Gaussian random variable, so we find it's mean and variance.

   $$E[X - Y] = E[X] - E[Y] = 1 - 2 = -1.$$

   $$\begin{aligned}\mathrm{Var}(X - Y) &= \mathrm{Cov}(X - Y, X - Y) = \mathrm{Var}(X) - 2\mathrm{Cov}(X, Y) + Var(Y) \\ &= \mathrm{Var}(X) - 2\rho\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)} + \mathrm{Var}(Y) = 4 + 8 + 16 = 28.\end{aligned}$$

   $$P\{X - Y \geq 5\} = P\left(\frac{X - Y + 1}{\sqrt{28}} \geq \frac{5 + 1}{\sqrt{28}}\right) = Q\left(\frac{6}{\sqrt{28}}\right) = Q\left(\frac{3}{\sqrt{7}}\right).$$

(c) Find $E[X|X+Y]$.
Solution: Since X and Y are jointly Gaussian, X and $X+Y$ are also jointly Gaussian.

$$\mathrm{Cov}(X, X+Y) = \mathrm{Var}(X)+\mathrm{Cov}(X,Y) = \mathrm{Var}(X)+\rho\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)} = 4-0.5\times 8 = 0.$$

Hence X and $X+Y$ are independent.

$$E[X|X+Y] = E[X] = 1.$$

(d) Find $E[Y^2|X=2]$.
Solution: Since X and Y are jointly Gaussian, $E[Y|X] = L^*(X)$, the MMSE linear estimator of Y given X. The standard formulas for $L^*(X)$ and the MSE σ_e^2 give the mean and variance of the conditional distribution of Y given X, which are combined to get the second moment of the conditional distribution:

$$E[Y|X=2] = L^*(2) = E[Y] + \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}(2-E[X]) = 2 + \frac{-4}{4}(2-1) = 1.$$
$$\mathrm{Var}(Y|X=2) = \mathrm{Var}(Y)(1-\rho^2) = 16\times\frac{3}{4} = 12.$$
$$E[Y^2|X=2] = \mathrm{Var}(Y|X=2) + (E[Y|X=2])^2 = 12+1 = 13.$$

9. **[14 points]** Suppose a new car costs 20 thousand dollars and the lifetime, T, of the car is exponentially distributed with *mean* 10 years. The time-average cost per year of the car, averaged over the lifetime of the car, in thousands of dollars per year, is thus given by $Y = \frac{20}{T}$.

(a) Find $E[Y]$.
Solution: The pdf of T is $f_T(t) = (0.1)e^{-(0.1)t}$ for $t \geq 0$ (and, of course, $f_T(t) = 0$ for $t < 0$). So by the LOTUS rule,

$$E[Y] = \int_0^\infty \frac{20}{t}(0.1)e^{-(0.1)t}dt = +\infty.$$

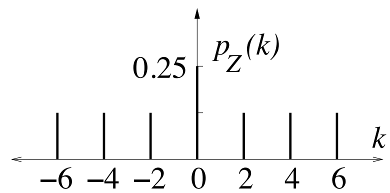
The integral is divergent because of the $1/t$ singularity at zero. This result may seem distressing to a customer. However, if T is very small, then the payment rate of $\frac{20}{T}$ only applies for a short period of time T. Also, manufacturers usually include a one-year warranty. Finally, the exponential distribution, corresponding to constant failure rate, may not accurately model the lifetime of a car.

(b) Find the pdf of Y, $f_Y(c)$, for all $c \geq 0$.
Solution: We note that $Y \geq 0$. For $c \geq 0$, $F_Y(c) = P\{\frac{20}{T} \leq c\} = P\{T \geq \frac{20}{c}\} = e^{-(0.1)20/c} = e^{-2/c}$. Differentiation yields $f_Y(c) = \frac{2}{c^2}e^{-2/c}$ for $c > 0$.

10. **[15 points]** Suppose X_1, X_2, X_3 are independent random variables such that for each i, $P\{X_i = 1\} = P\{X_i = -1\} = 0.5$. Let $Z = X_1 + 2X_2 + 3X_3$.

(a) Find and carefully sketch the pmf of Z. Be sure to label the sketch well.

Solution: We use "+" to denote value +1 and "-" to denote value -1, The tuple (X_1, X_2, X_3, Z) takes eight values with equal probability: $+++6, -++4, +-+2, --+0, ++-0, -+-(-2), +--(-4), ---(-6)$. So the possible values of Z are $\{-6,-4,-2,0,2,4,6\}$. Since $\{Z=0\}$ can happen two ways, the respective probabilities for the possible values of Z are $\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{2}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}$.



(b) Find $\mathrm{Var}(Z)$.

Solution: Since $E[X_i] = 0$ and $E[X_i^2] = 1$, observe that $\mathrm{Var}(X_i) = 1$. The variance of the sum of independent random variables is the sum of the variances, so $\mathrm{Var}(Z) = \mathrm{Var}(X_1) + 2^2\mathrm{Var}(X_1) + 3^2\mathrm{Var}(X_1) = 14$.

(c) Find $E[Z|Z \geq 0]$.

Solution: $P\{Z \geq 0\} = \frac{5}{8}$ and $P(Z = i|Z \geq 0)$ is $\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}$, for $i = 0, 2, 4, 6$, respectively. So $E[Z|Z \geq 0] = 0 \cdot \frac{2}{5} + 2 \cdot \frac{1}{5} + 4 \cdot \frac{1}{5} + 6 \cdot \frac{1}{5} = \frac{12}{5} = 2.4$.

11. [**12 points**]

(a) The first nine letters of the alphabet are randomly grouped into three piles of three letters each. What is the probability letters A and B end up in different piles?

Solution: We first find the probability A and B both end up in the first pile. There are $\binom{9}{3} = 84$ ways to choose 3 of 9 letters for the pile, and 7 ways to choose 3 of 9 letters that includes both A and B. So the probability the first pile contains both A and B is $7/84 = 1/12$. Similarly, the probability the second pile contains both A and B is $1/12$, and the probability the third pile contains both A and B is $1/12$. Since these three possibilities are mutually exclusive, the probability A and B are both in the same pile is $3/12$ or $1/4$. The probability that A and B will be in different piles is the complement: $3/4$.

ALTERNATIVELY, place A first. This leaves 8 places for B. Two of the places are in the same pile as A, while 6 are in other piles. So the desired probability is $6/8 = 3/4$.

(b) Three of the points below are chosen at random and connected by line segments. What is the probability a triangle (enclosing a region of positive area) is formed? (For full credit, express your final answer as a fraction in reduced form; no binomial coefficients should be included.)

Solution: Since $\binom{11}{3} = \frac{11 \cdot 10 \cdot 9}{3 \cdot 2 \cdot 1} = 165$, there are 165 ways to select the three points. A valid triangle is *not* formed if the points all come from the top row ($\binom{5}{3} = 10$

possibilities) or if they all come from the bottom row ($\binom{6}{3} = 20$ possibilities.) So the desired probability is $\frac{165-10-20}{165} = \frac{135}{165} = \frac{9}{11}$.

12. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) A network consists of a series connection of 50 links from node s to node t with failure probability of each link being equal to 0.01. Let F be the event that the network fails (i.e. at least one link fails).

TRUE	FALSE	
☐	☐	$P(F) = 1 - (0.01)^{50}$ if the links fail independently.
☐	☐	$P(F) \leq 0.5$ whether or not the links fail independently.

Solution: False,True

(b) Suppose A, B, and C are *pairwise* independent events, with probability 1/2 each.

TRUE	FALSE	
☐	☐	If A is independent of BC, then A, B, C are mutually independent.
☐	☐	$P(A\|B) = P(A\|C)$.
☐	☐	$P(ABC) \leq \frac{1}{8}$.

Solution: True,True,False

(c) Suppose X and Y are jointly continuous random variables with mean zero.

TRUE	FALSE	
☐	☐	$P\{X = Y\}$ must equal zero.
☐	☐	If $E[X\|Y] = E[Y\|X]$ then Y is a function of X.
☐	☐	If X and Y are independent then $\sin(X)$ and $\sin(Y)$ are uncorrelated.

Solution: True,False,True

(d) Suppose X and Y are jointly continuous random variables.

TRUE	FALSE	
☐	☐	If $\hat{E}[Y\|X] = 3X + 1$ then $\hat{E}[X\|Y] = \frac{1}{3}X - \frac{1}{3}$.
☐	☐	If $\hat{E}[Y\|X] = 1$, then X and Y are independent.

Solution: False, False.