University of Illinois Spring 2016

ECE 313: Conflict Final Exam

Tuesday, May 10, 2016 7 p.m. — 10 p.m.
Armory 101

1. [**14 points**] Consider a five-sided die, with equiprobable sides labeled $1, 2, 3, 4, 5$.

   (a) If you roll the die twenty times, what is the probability that for exactly five of the rolls, the outcome is in $\{2, 3\}$.
   Solution: In each roll, the event $A = \{\text{either 2 or 3 shows}\}$ has probability $\frac{2}{5}$. The rolls are mutually independent, so the number of times event A occurs in twenty rolls has the $Binomial(20, 2/5)$ distribution. Therefore, $P\{\text{either 2 or 3 shows five times in twenty rolls}\} = \binom{20}{5}\left(\frac{2}{5}\right)^5\left(\frac{3}{5}\right)^{15}$.

   (b) If you roll the die twenty times, and you know that the sixth roll is a 4, what is the probability that exactly five times after the sixth roll: the outcome is in $\{2, 3\}$?
   Solution: Each roll is independent of other rolls, hence the number of times event A occurs in the last fourteen rolls given that it did not occur on the sixth roll is $Binomial(14, 2/5)$. Therefore, $P\{\text{either 2 or 3 show five times in the last fourteen rolls}|\text{sixth roll is a 4}\} = \binom{14}{5}\left(\frac{2}{5}\right)^5\left(\frac{3}{5}\right)^9$.

   (c) If you know that the sixth roll is a 2, what is the expected number of additional rolls until the outcome is in $\{2, 3\}$ again?
   Solution: Each roll is independent of other rolls, hence the number of rolls until event A occurs after the sixth roll is $Geometric(2/5)$. Therefore, $E[\text{number of additional rolls after the sixth roll for either 2 or 3 to show}|\text{sixth roll is a 2}\} = \frac{1}{\frac{2}{5}} = \frac{5}{2}$.

2. [**14 points**] A random variable X is drawn from one of two possible distributions: H_1 : Poisson with parameter $\lambda = 1$ or H_0 : Poisson with parameter $\lambda = e$.
   You may need the fact $e \approx 2.7$.

   (a) If $X = 3$, which hypothesis does the maximum likelihood (ML) rule choose?
   Solution: The pmf of X under H_1 is given by $p_1(k) = \frac{e^{-1}}{k!}$ and the pmf of X under H_0 is given by $p_0(k) = \frac{e^{-e+k}}{k!}$. Thus, the likelihood ratio is $\Lambda(k) = \frac{p_1(k)}{p_0(k)} = e^{e-1-k}$. Since $\Lambda(3) = e^{e-4} < 1$, the ML decision for observation $X = 3$ is H_0.

   (b) If $X = 3$, what hypothesis does the MAP-rule choose if $\pi_1 = e^2\pi_0$?
   Solution: So $\frac{\pi_0}{\pi_1} = e^{-2}$. Since $\Lambda(3) = e^{e-4} > e^{-2}$, the MAP decision for observation $X = 3$ is H_1.

3. [**14 points**] Let T be an exponentially distributed random variable with parameter $\lambda = \ln 2$.

   (a) Find $P(T < 2)$. Simplify your answer as much as possible.
   Solution:
   $$P(T < 2) = 1 - e^{-2\lambda} = 1 - \frac{1}{4} = \frac{3}{4},$$
   where we used: $e^{-2\lambda} = (e^{-\lambda})^2 = (e^{-\ln 2})^2 = (\frac{1}{2})^2 = \frac{1}{4}$.

(b) Find $E(T|T > 2)$.

Solution: Due to the memoryless property,

$$E(T|T > 2) = 2 + \frac{1}{\lambda} = 2 + \frac{1}{\ln 2}.$$

4. [**14 points**] The NASA Long Duration Exposure Facility (LDEF) encountered an average of 10 impacts per day over a 5.7 year period. Suppose the timing of the impacts is well modeled by a Poisson process with rate parameter 10/day.

(a) What is the probability of exactly 100 impacts in a one week period?

Solution: The number of impacts in a week (seven days) has the Poisson distribution with mean 70. So the requested probability is $\frac{e^{-70}(70)^{100}}{100!}$.

(b) Given there are 60 impacts in the first three days, what is the conditional mean number of impacts in the first day?

Solution: Given 60 counts in the three day time period, the arrival times of the counts are uniformly distributed over the time period, so each would occur in the first day with probability 1/3. So the conditional mean for one day is 60/3=20.

(c) Given there are 60 impacts in the first three days, what is the conditional mean number of impacts in the first *week*?

Solution: There are 60 impacts in the first three days, and ten impacts are expected for each of the remaining four days of the week. This gives a total of $60 + 4 \cdot 10$, or 100, impacts for the week, given there were 60 impacts the first three days.

5. [**15 points**] Two random variables X and Y take values u and v, respectively, in the set $\{0, 1, 2, 3\}$.

(a) The table below partially gives the joint pmf and the marginal pmfs. It is known that: $P\{X = 3, Y \geq 2\} = 0.05$ and $P\{X = Y\} = 0.06$. Complete the table by filling in the seven underlined blanks.

		Y				
		$v = 0$	$v = 1$	$v = 2$	$v = 3$	$p_X(u)$
	$u = 0$	0.01	___	0.1	___	0.3
X	$u = 1$	0	0	___	0.1	0.2
	$u = 2$	0.09	0.01	0.05	0.05	0.2
	$u = 3$	0.05	0.2	___	___	___
	$p_Y(v)$	0.15	___	0.3	0.25	

Solution: $P\{X = Y\}$ is the sum of the diagonal entries of the joint pmf, so $p(3, 3) = 0$. Then the information $P\{X = 3, Y \geq 2\} = 0.05$ yields $p(3, 2) = 0.05$. Fill in the remainder of the blanks using the fact the marginal pmfs are given by the row or column sums of the joint pmf.

	$v=0$	$v=1$	$v=2$	$v=3$	$p_X(u)$
$u=0$	0.01	0.09	0.1	0.1	0.3
$u=1$	0.0	0.0	0.1	0.1	0.2
$u=2$	0.09	0.01	0.05	0.05	0.2
$u=3$	0.05	0.2	0.05	0.0	0.3
$p_Y(v)$	0.15	0.3	0.3	0.25	

(b) Given $Y=0$, what is the conditional probability $X=3$?
Solution: This part asks for $P(X=3|Y=2)=\frac{0.05}{0.15}=1/3$.

(c) The joint pmf of W and Z and its marginals are shown below. Determine the conditional probability: $P\{Z \text{ is even}|W \text{ is odd}\}$.

			Z			
		$v=0$	$v=1$	$v=2$	$v=3$	$p_W(u)$
	$u=0$	0.01	0.09	0.1	0.05	0.25
W	$u=1$	0.0	0.05	0.1	0.05	0.2
	$u=2$	0.09	0.01	0.00	0.05	0.15
	$u=3$	0.1	0.2	0.05	0.05	0.4
	$p_Z(v)$	0.2	0.35	0.25	0.20	

Solution: This is $P\{Z=0,2|W=1,3\}=\frac{P\{Z=0,2,W=1,3\}}{P\{W=1,3\}}=\frac{0.0+0.1+0.1+0.05}{0.2+0.4}=5/12$.

6. [**18 points**] Let X and Y be random variables with joint pdf $f_{X,Y}(u,v)=\frac{1}{6}$ for $0\le u\le 3, v\in\{(1,2)\cup(3,4)\}$, and zero else.

(a) Are X and Y independent? Indicate why or why not.
Solution: If $X\sim Uniform(0,3)$ and $Y\sim Uniform\left((1,2)\cup(3,4)\right)$, then $f_{X,Y}(u,v)=f_X(u)f_Y(v)$ everywhere on the plane. Hence they are independent.

(b) Obtain the marginal $f_Y(v)$ for all v.
Solution: $Y\sim Uniform\left((1,2)\cup(3,4)\right)$, therefore $f_y(v)=\frac{1}{2}$ for $v\in\{(1,2)\cup(3,4)\}$ and zero else.

(c) Obtain the conditional pdf $f_{X|Y}(u|v)$ for all u,v.
Solution: From part (a), we know that X and Y are independent, hence $X\sim Uniform(0,3)$ for all $v\in\{(1,2)\cup(3,4)\}$ and it is undefined else; that is, if $v\in\{(1,2)\cup(3,4)\}$, $f_{X|Y}(u|v)=\frac{1}{3}$ for $u\in(0,3)$ and zero else.

(d) Let $Z=X^2+Y$. Obtain $P\{Z\le 2\}$.
Solution:

$$P\{Z\le 2\}=\int_1^2\int_0^{\sqrt{2-v}}\frac{1}{6}dudv=\frac{1}{6}\int_1^2\sqrt{2-v}dv=\frac{1}{9}$$

7. [**20 points**] Continuous random variables X,Y are uniformly distributed over regions $\mathcal{C}_a$,$\mathcal{C}_b$,$\mathcal{C}_c$ and $\mathcal{C}_d$ which are shown in Figures (a),(b),(c) and (d), respectively. In each of the cases find the means μ_X, μ_Y and the covariance, $\mathrm{Cov}(X,Y)$. Numerical answers are expected.

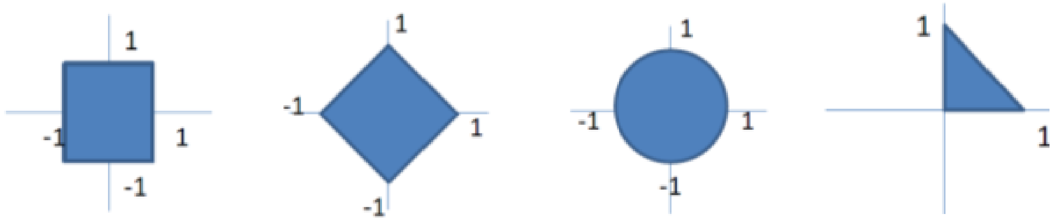


Figure a Figure b Figure c Figure d

Solution: In each of Figures (a), (b), and (c), the regions are symmetric around the origin for both X and Y dimensions. Hence $\mu_X = \mu_Y = 0$. For these regions we also have $E[XY] = 0$, again due to the symmetry around the origin. Thus $\text{Cov}(X, Y) = 0$. For Figure (d), the scenario requires explicit computations. The pdf over the triangular region has value 2, so

$$\mu_X = 2\int_0^1 \int_0^{1-u} u \, dv \, du = 2\int_0^1 u(1-u) \, du = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}.$$

Due to symmetry, we have $\mu_X = \mu_Y$. Similarly,

$$E[XY] = 2\int_0^1 \int_0^{1-u} uv \, dv \, du = 2\int_0^1 \frac{u}{2}(1-u)^2 \, du = \frac{1}{12}.$$

Thus $\text{Cov}(X, Y) = \frac{1}{12} - \frac{1}{9} = -\frac{1}{36}$.

8. [**20 points**] Suppose X and Y have a bivariate Gaussian joint distribution with $E[X] = 1$, $E[Y] = 2$, $\text{Var}(X) = 4$, $\text{Var}(Y) = 16$, and the correlation coefficient $\rho = -0.5$.

   (a) Are $X - Y$ and $X + Y$ independent? Why or why not?
   Solution: Since X and Y are jointly Gaussian, $X - Y$ and $X + Y$ are jointly Gaussian too. Hence they are independent if and only if they are uncorrelated.

   $$\text{Cov}(X - Y, X + Y) = \text{Var}(X) - \text{Var}(Y) = 4 - 16 = -12 \neq 0,$$

   so $X - Y$ and $X + Y$ are *not* independent.

   (b) Find $P\{X - Y \geq 5\}$.
   Solution: $X - Y$ is a Gaussian random variable, so we find it's mean and variance.

   $$E[X - Y] = E[X] - E[Y] = 1 - 2 = -1.$$

   $$\begin{aligned} \text{Var}(X - Y) &= \text{Cov}(X - Y, X - Y) = \text{Var}(X) - 2\text{Cov}(X, Y) + Var(Y) \\ &= \text{Var}(X) - 2\rho\sqrt{\text{Var}(X)\text{Var}(Y)} + \text{Var}(Y) = 4 + 8 + 16 = 28. \end{aligned}$$

   $$P\{X - Y \geq 5\} = P\left(\frac{X - Y + 1}{\sqrt{28}} \geq \frac{5 + 1}{\sqrt{28}}\right) = Q\left(\frac{6}{\sqrt{28}}\right) = Q\left(\frac{3}{\sqrt{7}}\right).$$

(c) Find $E[X|X+Y]$.
Solution: Since X and Y are jointly Gaussian, X and $X+Y$ are also jointly Gaussian.

$$\mathrm{Cov}(X, X+Y) = \mathrm{Var}(X)+\mathrm{Cov}(X,Y) = \mathrm{Var}(X)+\rho\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)} = 4-0.5\times 8 = 0.$$

Hence X and $X+Y$ are independent.

$$E[X|X+Y] = E[X] = 1.$$

(d) Let $Z = aX + bY$ where $a > 0$. For what numerical values of a and b is Z standard normal?
Solution: We have two equation ($\mu_Z = 0$ and $\sigma_Z^2 = 1$) and two unknowns. The equations become:

$$\mu_Z = a\mu_X + b\mu_Y = a + 2b = 0$$
$$\sigma_Z^2 = 4a^2 + 16b^2 - \frac{1}{2} \cdot 2 \cdot 4 \cdot 2ab = 4a^2 + 16b^2 - 8ab$$

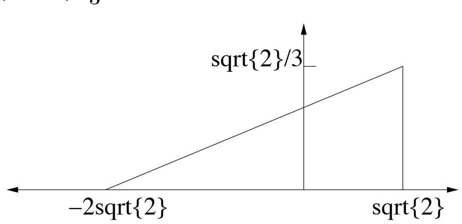
Setting $a = -2b$ in the second equation gives $16b^2 + 16b^2 + 16b^2 = 1$ so $b = \pm\frac{1}{\sqrt{48}} = \pm\frac{1}{4\sqrt{3}}$. So there is one solution with $a > 0$: $(a, b) = (\frac{1}{2\sqrt{3}}, -\frac{1}{4\sqrt{3}})$.

9. **[14 points]** Suppose the pdf of X is $f_X(u) = cu$ for $0 \le u \le 1$ and $f_X(u) = 0$ for other values of u.

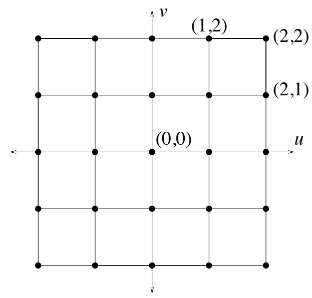
(a) Find c.
Solution: The area under the pdf is $c/2$ which should equal one, so $c = 2$.

(b) Let Y be the standardized version of X so it has the form $Y = (X - a)/b$. Find the numerical values of a and b.
Solution: In general, $a = E[X]$ and b is equal to σ, the standard deviation of X. $a = E[X] = \int_{-\infty}^{\infty} f_X(u) = \int_0^1 u \cdot 2udu = \frac{2u^3}{3}\Big|_0^1 = \frac{2}{3}$. $E[X^2] = \int_0^1 u^2 \cdot 2udu = \frac{2u^4}{4}\Big|_0^1 = \frac{1}{2}$. Thus, $\sigma^2 = \mathrm{Var}(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$. Thus, $b = \sigma = \frac{1}{\sqrt{18}} = \frac{1}{3\sqrt{2}}$.

(c) Sketch the pdf of Y. Carefully label the axes.
Solution: The pdf of Y has the same shape as the pdf of X; it just needs to be translated and scaled. Just subtracting 2/3 from X would cause the pdf to shift left by 2/3, giving a triangle with base $[-2/3, 1/3]$. Division of (X-2/3) by σ, or eqivalently, multiplication by $\frac{1}{\sigma}$, causes the base to expand by the factor $\frac{1}{\sigma}$, to the interval $\left[-2\sqrt{2}, \sqrt{2}\right]$. The new height is $c\sigma = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}$. To double check, the area under the pdf of Y is $\frac{1}{2}(3\sqrt{2})\frac{\sqrt{2}}{3} = 1$.



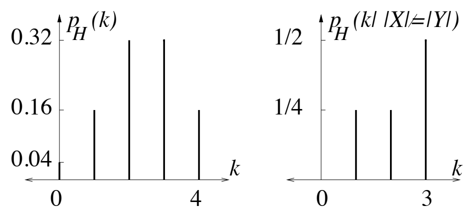
10. [**15 points**] A graph with 25 vertices and unit length edges is overlaid on the $u-v$ plane as shown.



The center vertex has coordinates $(0,0)$, and the coordinates of three other vertices are shown. Let X and Y be independent random variables, each taking values in $\{-2,-1,0,1,2\}$ with equal probabilities. Thus, (X,Y) represents the coordinates of a graph vertex selected uniformly at random. Let H denote the graph distance (minimum number of hops) between (X,Y) and the center vertex.

(a) Find and carefully sketch the pmf of H. Be sure to label the sketch well.

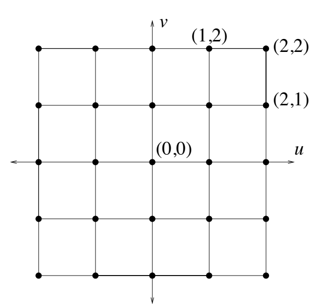
Solution: There are 1,4,8,8, or 4 vertices with graph distance 0, 1, 2, 3, or 4, respectively, from the central vertex. Thus, H takes the possible values 0, 4, 8, 8, or 4 with respective probabilities 0.04, 0.16. 0.32, 0.32, or 0.16, respectively.



(b) Find the pmf of H, given $|X| \neq |Y|$.

Solution: There are 16 vertices with $|u| \neq |v|$, namely, their coordinates are: (0,1), (1,0), (0,-1), (-1,0): one hop from central vertex),
(0,2), (2,0),(0,-2), (-2,0) : two hops from central vertex,
(1,2),(2,1),(-1,2),(2,-1),(1,-2),(-2,1),(-1,-2),(-2,-1): three hops from central vertex.
Given $X \neq Y$, the conditional probability (X,Y) is equal to any one of these 16 vertices is 1/16. Thus, given $|X| \neq |Y|$, H takes values 1, 2, or 3 with conditional probability $\frac{4}{16}$, $\frac{4}{16}$, or $\frac{8}{16}$, respectively,

(c) Let D denote the Euclidean distance from (X,Y) to the center vertex (i.e. $D^2 = X^2 + Y^2$). Find $E[D^2]$.



Solution: X^2 takes value 0,1, or 4 with probability $\frac{1}{5}$, $\frac{2}{5}$, or $\frac{2}{5}$, respectively. Thus $E[X^2] = 0 \cdot \frac{1}{5} + 1 \cdot \frac{2}{5} + 4 \cdot \frac{2}{5} = 2$. Similarly, $E[Y^2] = 2$. So $E[D^2] = E[X^2] + E[Y^2] = 4$.

11. [**12 points**]

(a) Corey and Tony are friends on the same team. There are 8 players on the team. A starting lineup of five players is selected at random. What is the probability that Corey or Tony or both are in the lineup?

Solution: First, we count the number of possible lineups: 8 choose 5 which equals 56. Lineups that *exclude* both Corey and Tony can be done in 6 choose 5 ways, which equals 6. So the probability that Corey or Tony or both are in the lineup is equal to (56-6)/56 = 50/56 = 25/28.

(b) Three points below are chosen at random connected by lines. What is the probability a triangle (enclosing a region with positive area) is formed?

Solution: The total number of ways in which 3 points can chosen is 11 choose 3 ways, which is equal to 165. But there are 5 choose 3 =10 ways to select three points on the top row and 6 choose 3 = 20 ways to select three points from the bottom row. In each of these two cases, a triangle is not formed. So the desired probability is 135/165 = 9/11.

12. [**30 points**] (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Consider a binary hypothesis testing problem with some known prior distribution (π_0, π_1). Let $p_{e,MAP}$ and $p_{e,ML}$ be the average probability of error of the MAP and ML rules, respectively.

TRUE	FALSE	
☐	☐	$p_{e,MAP} \leq p_{e,ML}$.
☐	☐	$p_{false-alarm} + p_{miss} \leq 1$ for the ML rule.

Solution: True,True

(b) Let Φ denote the CDF of the standard normal distribution and let Q denote the complementary CDF of the same distribution.

TRUE	FALSE	
☐	☐	$Q(3) = 1 - Q(-3)$
☐	☐	$\Phi(1) + \Phi(2) = \Phi(3)$
☐	☐	$\Phi(4) - \Phi(3) = Q(3) - Q(4)$.

Solution: True, False, True

(c) Suppose that T is a nonnegative random variable with failure rate function $(h(t) : t \geq 0)$.

TRUE	FALSE	
☐	☐	If h is constant then T is exponentially distributed.
☐	☐	If the failure rate of a random variable S is $2h(t)$ then it must be that $E[T] = 2E[S]$.

Solution: True, False

(d) Suppose X and Y are jointly continuous random variables with variance one.

TRUE	FALSE	
☐	☐	$E[(Y - E[Y\|X])^2] \leq \text{Var}(Y)$.
☐	☐	If $E[X\|Y] = Y$ then $\widehat{E}[X\|Y] = Y$.
☐	☐	If X and Y are uncorrelated then X^2 and Y^2 are uncorrelated.

Solution: True,True,False