University of Illinois Spring 2016

ECE 313: Conflict Final Exam

Tuesday, May 10, 2016 7 p.m. — 10 p.m.
Armory 101

Name: (in BLOCK CAPITALS) ______________________________

NetID: ______________________________

Signature: ______________________________

Section:
☐ A, 9:00 a.m. ☐ B, 10:00 a.m. ☐ C, 11:00 a.m. ☐ D, 1:00 p.m. ☐ E, 2:00 p.m.

Instructions

This exam is closed book and closed notes except that two 8.5”×11” sheets of notes is permitted: both sides may be used. No electronic equipment (cell phones, etc.) allowed.

The exam consists of 12 problems worth a total of 200 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading

1. 14 points ________
2. 14 points ________
3. 14 points ________
4. 14 points ________
5. 15 points ________
6. 18 points ________
7. 20 points ________
8. 20 points ________
9. 14 points ________
10. 15 points ________
11. 12 points ________
12. 30 points ________

Total (200 points) ________

1. [**14 points**] Consider a five-sided die, with equiprobable sides labeled $1, 2, 3, 4, 5$.

   (a) If you roll the die twenty times, what is the probability that for exactly five of the rolls, the outcome is in $\{2, 3\}$.

   (b) If you roll the die twenty times, and you know that the sixth roll is a 4, what is the probability that exactly five times after the sixth roll: the outcome is in $\{2, 3\}$?

   (c) If you know that the sixth roll is a 2, what is the expected number of additional rolls until the outcome is in $\{2, 3\}$ again?

2. [**14 points**] A random variable X is drawn from one of two possible distributions: H_1 : Poisson with parameter $\lambda = 1$ or H_0 : Poisson with parameter $\lambda = e$.
You may need the fact $e \approx 2.7$.

(a) If $X = 3$, which hypothesis does the maximum likelihood (ML) rule choose?

(b) If $X = 3$, what hypothesis does the MAP-rule choose if $\pi_1 = e^2 \pi_0$?

3. [**14 points**] Let T be an exponentially distributed random variable with parameter $\lambda = \ln 2$.

(a) Find $P(T < 2)$. Simplify your answer as much as possible.

(b) Find $E(T|T > 2)$.

4. [**14 points**] The NASA Long Duration Exposure Facility (LDEF) encountered an average of 10 impacts per day over a 5.7 year period. Suppose the timing of the impacts is well modeled by a Poisson process with rate parameter 10/day.

   (a) What is the probability of exactly 100 impacts in a one week period?

   (b) Given there are 60 impacts in the first three days, what is the conditional mean number of impacts in the first day?

   (c) Given there are 60 impacts in the first three days, what is the conditional mean number of impacts in the first *week*?

5. **[15 points]** Two random variables X and Y take values u and v, respectively, in the set $\{0, 1, 2, 3\}$.

(a) The table below partially gives the joint pmf and the marginal pmfs. It is known that: $P\{X = 3, Y \geq 2\} = 0.05$ and $P\{X = Y\} = 0.06$. Complete the table by filling in the seven underlined blanks.

		Y				
		$v = 0$	$v = 1$	$v = 2$	$v = 3$	$p_X(u)$
	$u = 0$	0.01	___	0.1	___	0.3
X	$u = 1$	0	0	___	0.1	0.2
	$u = 2$	0.09	0.01	0.05	0.05	0.2
	$u = 3$	0.05	0.2	___	___	___
	$p_Y(v)$	0.15	___	0.3	0.25	

(b) Given $Y = 0$, what is the conditional probability $X = 3$?

(c) The joint pmf of W and Z and its marginals are shown below. Determine the conditional probability: $P\{Z \text{ is even}|W \text{ is odd}\}$.

		Z				
		$v = 0$	$v = 1$	$v = 2$	$v = 3$	$p_W(u)$
	$u = 0$	0.01	0.09	0.1	0.05	0.25
W	$u = 1$	0.0	0.05	0.1	0.05	0.2
	$u = 2$	0.09	0.01	0.00	0.05	0.15
	$u = 3$	0.1	0.2	0.05	0.05	0.4
	$p_Z(v)$	0.2	0.35	0.25	0.20	

6. **[18 points]** Let X and Y be random variables with joint pdf $f_{X,Y}(u,v) = \frac{1}{6}$ for $0 \leq u \leq 3, v \in \{(1,2) \cup (3,4)\}$, and zero else.

(a) Are X and Y independent? Indicate why or why not.

(b) Obtain the marginal $f_Y(v)$ for all v.

(c) Obtain the conditional pdf $f_{X|Y}(u|v)$ for all u, v.

(d) Let $Z = X^2 + Y$. Obtain $P\{Z \leq 2\}$.

7. [**20 points**] Continuous random variables X, Y are uniformly distributed over regions $\mathcal{C}_a$,$\mathcal{C}_b$,$\mathcal{C}_c$ and $\mathcal{C}_d$ which are shown in Figures (a),(b),(c) and (d), respectively. In each of the cases find the means μ_X, μ_Y and the covariance, $\text{Cov}(X, Y)$. Numerical answers are expected.

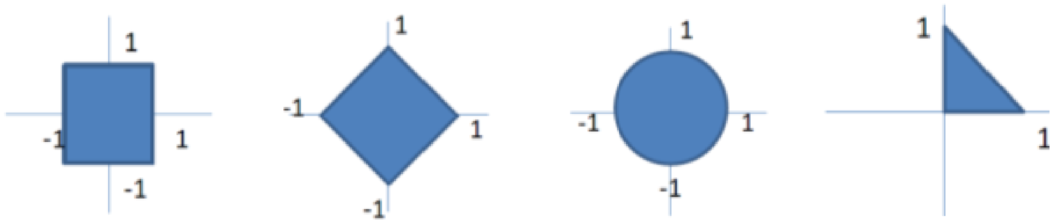


Figure a Figure b Figure c Figure d

8. **[20 points]** Suppose X and Y have a bivariate Gaussian joint distribution with $E[X] = 1$, $E[Y] = 2$, $\text{Var}(X) = 4$, $\text{Var}(Y) = 16$, and the correlation coefficient $\rho = -0.5$.

   (a) Are $X - Y$ and $X + Y$ independent? Why or why not?

   (b) Find $P\{X - Y \geq 5\}$.

   (c) Find $E[X|X + Y]$.

   (d) Let $Z = aX + bY$ where $a > 0$. For what numerical values of a and b is Z standard normal?

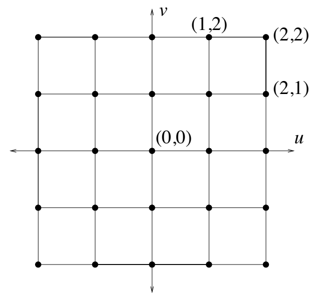
9. [**14 points**] Suppose the pdf of X is $f_X(u) = cu$ for $0 \leq u \leq 1$ and $f_X(u) = 0$ for other values of u.

(a) Find c.

(b) Let Y be the standardized version of X so it has the form $Y = (X - a)/b$. Find the numerical values of a and b.

(c) Sketch the pdf of Y. Carefully label the axes.

10. [**15 points**] A graph with 25 vertices and unit length edges is overlaid on the $u-v$ plane as shown.

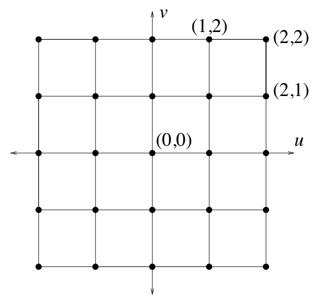


The center vertex has coordinates $(0, 0)$, and the coordinates of three other vertices are shown. Let X and Y be independent random variables, each taking values in $\{-2, -1, 0, 1, 2\}$ with equal probabilities. Thus, (X, Y) represents the coordinates of a graph vertex selected uniformly at random. Let H denote the graph distance (minimum number of hops) between (X, Y) and the center vertex.

(a) Find and carefully sketch the pmf of H. Be sure to label the sketch well.

(b) Find the pmf of H, given $|X| \neq |Y|$.

(c) Let D denote the Euclidean distance from (X, Y) to the center vertex (i.e. $D^2 = X^2 + Y^2$). Find $E[D^2]$.



11. [**12 points**]

(a) Corey and Tony are friends on the same team. There are 8 players on the team. A starting lineup of five players is selected at random. What is the probability that Corey or Tony or both are in the lineup?

(b) Three points below are chosen at random connected by lines. What is the probability a triangle (enclosing a region with positive area) is formed?

12. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Consider a binary hypothesis testing problem with some known prior distribution (π_0, π_1). Let $p_{e,MAP}$ and $p_{e,ML}$ be the average probability of error of the MAP and ML rules, respectively.

TRUE	FALSE	
☐	☐	$p_{e,MAP} \leq p_{e,ML}$.
☐	☐	$p_{false-alarm} + p_{miss} \leq 1$ for the ML rule.

(b) Let Φ denote the CDF of the standard normal distribution and let Q denote the complementary CDF of the same distribution.

TRUE	FALSE	
☐	☐	$Q(3) = 1 - Q(-3)$
☐	☐	$\Phi(1) + \Phi(2) = \Phi(3)$
☐	☐	$\Phi(4) - \Phi(3) = Q(3) - Q(4)$.

(c) Suppose that T is a nonnegative random variable with failure rate function $(h(t) : t \geq 0)$.

TRUE	FALSE	
☐	☐	If h is constant then T is exponentially distributed.
☐	☐	If the failure rate of a random variable S is $2h(t)$ then it must be that $E[T] = 2E[S]$.

(d) Suppose X and Y are jointly continuous random variables with variance one.

TRUE	FALSE	
☐	☐	$E[(Y - E[Y\|X])^2] \leq \mathrm{Var}(Y)$.
☐	☐	If $E[X\|Y] = Y$ then $\widehat{E}[X\|Y] = Y$.
☐	☐	If X and Y are uncorrelated then X^2 and Y^2 are uncorrelated.