University of Illinois Spring 2016

ECE 313: Final Exam

Monday, May 9, 2016 7 p.m. — 10 p.m.
ECEB 1002 - ECEB 1013

Name: (in BLOCK CAPITALS) ________________________________

NetID: ________________________________

Signature: ________________________________

Section:

☐ A, 9:00 a.m. ☐ B, 10:00 a.m. ☐ C, 11:00 a.m. ☐ D, 1:00 p.m. ☐ E, 2:00 p.m.

Instructions

This exam is closed book and closed notes except that two 8.5”×11” sheets of notes is permitted: both sides may be used. No electronic equipment (cell phones, etc.) allowed.

The exam consists of 12 problems worth a total of 200 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading

1. 14 points	______
2. 14 points	______
3. 14 points	______
4. 14 points	______
5. 15 points	______
6. 18 points	______
7. 20 points	______
8. 20 points	______
9. 14 points	______
10. 15 points	______
11. 12 points	______
12. 30 points	______
Total (200 points)	______

1. [**14 points**] Consider a five-sided die, with equiprobable sides labeled $1, 2, 3, 4, 5$.

   (a) If you roll the die twenty times, what is the probability that for exactly five of the rolls, the outcome is in $\{2, 3\}$?

   (b) If you roll the die twenty times, given that the sixth roll is a 4, what is the probability that for exactly five of the twenty rolls, the outcome is in $\{2, 3\}$?

   (c) If you know that the sixth roll is a 2, what is the probability that the next time that either a 2 or 3 shows is the fifteenth roll?

2. [**14 points**] A random variable X is drawn from one of two possible distributions:
H_1 : Poisson with parameter $\lambda = 1$ or H_0 : Poisson with parameter $\lambda = e$.
You may need the fact $e \approx 2.7$.

(a) If $X = 3$, which hypothesis does the maximum likelihood (ML) rule choose?

(b) If $X = 3$, what hypothesis does the MAP-rule choose if $\pi_1 = e^2 \pi_0$?

3. [**14 points**] Suppose $S = X_1 + X_2 + X_3$ where X_1, X_2, X_3 are mutually independent and X_i has the Bernoulli distribution with parameter $p_i = \frac{i}{5}$ for $i \in \{1, 2, 3\}$.

   (a) Find $P(S = 1)$.

   (b) Find $P(X_1 = 1 | S = 1)$.

4. [**14 points**] The NASA Long Duration Exposure Facility (LDEF) encountered an average of 10 impacts per day over a 5.7 year period. Suppose the timing of the impacts is well modeled by a Poisson process with rate parameter 10/day.

   (a) What is the probability of exactly 100 impacts in a one week period?

   (b) Given there are 60 impacts in the first three days, what is the conditional mean number of impacts in the first day?

   (c) Given there are 60 impacts in the first three days, what is the conditional mean number of impacts in the first *week*?

5. **[15 points]** Two random variables X and Y take values u and v, respectively, in the set $\{0, 1, 2, 3\}$.

(a) The table below partially gives the joint pmf and the marginal pmfs. It is known that: $P\{X = 3, Y \geq 2\} = 0.05$ and $P\{X = Y\} = 0.06$. Complete the table by filling in the seven underlined blanks.

		Y				
		$v = 0$	$v = 1$	$v = 2$	$v = 3$	$p_X(u)$
	$u = 0$	0.01	___	0.1	___	0.3
X	$u = 1$	0	0	___	0.1	0.2
	$u = 2$	0.09	0.01	0.05	0.05	0.2
	$u = 3$	0.05	0.2	___	___	___
	$p_Y(v)$	0.15	___	0.3	0.25	

(b) Given $Y = 0$, what is the conditional probability $X = 3$?

(c) The joint pmf of W and Z and its marginals are shown below. Determine the conditional probability: $P\{Z \text{ is even} | W \text{ is odd}\}$.

		Z				
		$v = 0$	$v = 1$	$v = 2$	$v = 3$	$p_W(u)$
	$u = 0$	0.01	0.09	0.1	0.05	0.25
W	$u = 1$	0.0	0.05	0.1	0.05	0.2
	$u = 2$	0.09	0.01	0.00	0.05	0.15
	$u = 3$	0.1	0.2	0.05	0.05	0.4
	$p_Z(v)$	0.2	0.35	0.25	0.20	

6. **[18 points]** Let X and Y be random variables with joint pdf $f_{X,Y}(u,v) = \frac{1}{2}e^{-u}$ for $u \geq 0, v \in (1,2) \cup (3,4)$, and zero else.

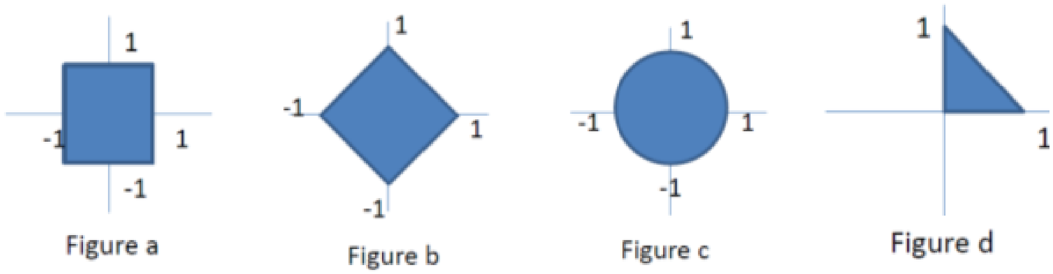
(a) Are X and Y independent? Indicate why or why not.

(b) Obtain the marginal $f_X(u)$ for all u.

(c) Obtain the conditional pdf $f_{Y|X}(v|u)$ for all u, v.

(d) Let $Z = X + Y$. Obtain $P\{Z \leq 3\}$.

7. [**20 points**] Continuous random variables X, Y are uniformly distributed over regions $\mathcal{C}_a$,$\mathcal{C}_b$,$\mathcal{C}_c$ and $\mathcal{C}_d$ which are shown in Figures (a),(b),(c) and (d), respectively. In each of the cases find the means μ_X, μ_Y and the covariance, $\text{Cov}(X, Y)$. Numerical answers are expected.



8. **[20 points]** Suppose X and Y have a bivariate Gaussian joint distribution with $E[X] = 1$, $E[Y] = 2$, $\text{Var}(X) = 4$, $\text{Var}(Y) = 16$, and the correlation coefficient $\rho = -0.5$.

   (a) Are $X - Y$ and $X + Y$ independent? Why or why not?

   (b) Find $P\{X - Y \geq 5\}$.

   (c) Find $E[X|X + Y]$.

   (d) Find $E[Y^2|X = 2]$.

9. **[14 points]** Suppose a new car costs 20 thousand dollars and the lifetime, T, of the car is exponentially distributed with *mean* 10 years. The time-average cost per year of the car, averaged over the lifetime of the car, in thousands of dollars per year, is thus given by $Y = \frac{20}{T}$.

   (a) Find $E[Y]$.

   (b) Find the pdf of Y, $f_Y(c)$, for all $c \geq 0$.

10. **[15 points]** Suppose X_1, X_2, X_3 are independent random variables such that for each i, $P\{X_i = 1\} = P\{X_i = -1\} = 0.5$. Let $Z = X_1 + 2X_2 + 3X_3$.

(a) Find and carefully sketch the pmf of Z. Be sure to label the sketch well.

(b) Find $\mathrm{Var}(Z)$.

(c) Find $E[Z|Z \geq 0]$.

11. [**12 points**]

(a) The first nine letters of the alphabet are randomly grouped into three piles of three letters each. What is the probability letters A and B end up in different piles?

(b) Three of the points below are chosen at random and connected by line segments. What is the probability a triangle (enclosing a region of positive area) is formed? (For full credit, express your final answer as a fraction in reduced form; no binomial coefficients should be included.)

12. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) A network consists of a series connection of 50 links from node s to node t with failure probability of each link being equal to 0.01. Let F be the event that the network fails (i.e. at least one link fails).

TRUE	FALSE	
☐	☐	$P(F) = 1 - (0.01)^{50}$ if the links fail independently.
☐	☐	$P(F) \leq 0.5$ whether or not the links fail independently.

(b) Suppose A, B, and C are *pairwise* independent events, with probability 1/2 each.

TRUE	FALSE	
☐	☐	If A is independent of BC, then A, B, C are mutually independent.
☐	☐	$P(A\|B) = P(A\|C)$.
☐	☐	$P(ABC) \leq \frac{1}{8}$.

(c) Suppose X and Y are jointly continuous random variables with mean zero.

TRUE	FALSE	
☐	☐	$P\{X = Y\}$ must equal zero.
☐	☐	If $E[X\|Y] = E[Y\|X]$ then Y is a function of X.
☐	☐	If X and Y are independent then $\sin(X)$ and $\sin(Y)$ are uncorrelated.

(d) Suppose X and Y are jointly continuous random variables.

TRUE	FALSE	
☐	☐	If $\hat{E}[Y\|X] = 3X + 1$ then $\hat{E}[X\|Y] = \frac{1}{3}X - \frac{1}{3}$.
☐	☐	If $\hat{E}[Y\|X] = 1$, then X and Y are independent.