University of Illinois Spring 2016

ECE 313: Hour Exam I

Wednesday, March 2, 2016
7:00 p.m. — 8:15 p.m.
Aa-Hs in DCL 1320,
Ht-Lf in ECEB 1013,
Lg-Np in ECEB 1015,
Nq-Zz in ECEB 1002

1. **[18 points]** Consider three events, A, B, C in a probability space. Let $P(AB) = 0.25$, $P(AB^c) = 0.25$. It is known that A and B are independent and that B^c and C are mutually exclusive. Find numerical values for the quantities below, and show your work.

   (a) Obtain $P(B)$.
   Solution: Since A is the union of the mutually exclusive events AB and AB^c, $P(A) = P(AB) + P(AB^c) = 0.5$. A and B are independent, hence $P(AB) = P(A)P(B)$, so that $P(B) = \frac{P(AB)}{P(A)} = 0.5$.

   (b) Obtain $P(A^cB^cC)$.
   Solution: B^c and C are mutually exclusive, so $A^cB^cC = \emptyset$. Therefore, $P(A^cB^cC) = 0$.

   (c) Obtain $P(A^cB^cC^c)$.
   Solution: (A good approach is to use a Karnaugh map. Start with events A and B. Then overlay event C using the assumption C does not overlap B^c.) From part (a) we see $P(A^cB^c) = P(A^c)P(B^c) = 0.25$. Since B^c and C are mutually exclusive, it follows that $B^c \subset C^c$. Hence $A^cB^cC^c = A^cB^c$ so $P(A^cB^cC^c) = 0.25$.

2. **[17 points]** Temperature X in degrees F corresponds to temperature $(X - 32) * (5/9)$ in degrees C. For example, 50 °F is equivalent to 10 °C. According to www.climatestations.com, the standard deviation of the daily temperature (in Chicago, years 1872-2008) is 6 °F in July and 12 °F in January.

   (a) What is the standard deviation of the daily temperature in July, measured in °C? Give the numerical value and show your work.
   Solution: Let $Y = (X - 32) * (5/9)$. Since standard deviation is not effected by adding constants to random variables and scales linearly, $\sigma_Y = (5/9)\sigma_X = (5/9) * 6 = 30/9 = 10/3$.

   (b) What is the *variance* of the daily temperature in July, if temperature is measured in °F. Give both the numerical value and the units of the variance.
   Solution: The variance is the standard deviation squared, so the variance for July is 36 $(^\circ F)^2$. That is, 36 measured in degrees F squared.

   (c) What is the ratio of the variance of the daily temperature in January to the variance of the daily temperature in July, both measured in °C. Give the numerical answer and briefly explain.
   Solution: The standard deviation is twice as large in January as in July, so the variance in January is **four** times the variance in July (for both variances defined using the same temperature units).

3. **[21 points]** Suppose two fair dice are rolled. Define the following three events:

   - D is "doubles" (the two numbers rolled are equal)
   - S is "the sum is seven"

- A is "the product is even."

(a) Which pair of events (from among D, S and A) are mutually exclusive?
Solution: D and S are mutually exclusive. $D = \{11, 22, 33, 44, 55, 66\}$ which has no outcomes in common with $S = \{16, 25, 34, 43, 52, 61\}$.

(b) Find $P(D|A)$. Show your work!
Solution: $A^c = \{11, 13, 15, 31, 33, 35, 51, 53, 55\}^c$ so $P(A) = 27/36$. Also, $P(AD) = P\{22, 44, 66\} = 3/36$. Therefore, $P(D|A) = (3/36)/(27/36) = 1/9$.

(c) Find $P(A|S)$. Show your work!
Solution: Every element of $S = \{16, 25, 34, 43, 52, 61\}$ is in A. That is, $AS = S$. So $P(AS) = P(S)$. Thus, $P(A|S) = P(AS)/P(S) = (1/6)/(1/6) = 1$.

4. **[16 points]** Suppose X is Binomial(n, p) under hypothesis H_0 and Binomial$(n, 1-p)$ under hypothesis H_1, such that $p < 0.5$ and n is odd.

(a) Given $X = k$ for some fixed number k, derive the ML rule to decide on the underlying hypothesis. *Hint*: your decision will depend on k.
Solution: The likelihood under H_0 is $\binom{n}{k}p^k(1-p)^{n-k}$ while the likelihood under H_1 is $\binom{n}{k}p^{n-k}(1-p)^k$. The ratio of these two likelihoods is $\Lambda(k) = \frac{p^{n-k}(1-p)^k}{p^k(1-p)^{n-k}} = \left(\frac{p}{1-p}\right)^{n-2k}$. Since $p < 0.5$, the ratio is less than one if $k < \frac{n}{2}$ and greater than one if $k > \frac{n}{2}$. So the ML rule is given by:

$$\text{ML rule:} \quad \text{If } k < \frac{n}{2} \text{ output } H_0; \qquad \text{If } k > \frac{n}{2} \text{ output } H_1.$$

(b) What is the probability of missed detection for $n = 5$ and $p = \frac{1}{3}$?
Solution: $p_{miss} = P(X \leq 2|H_1) = \sum_{k=0}^{2}\binom{5}{k}\left(\frac{1}{3}\right)^{5-k}\left(\frac{2}{3}\right)^k = \frac{51}{243} = \frac{17}{81}$.

5. **[16 points]** There are two drawers. The left drawer contains two green socks, three red socks and five orange socks. The right drawer contains two red socks and two orange socks. A sock is randomly picked from the left drawer and put into the right drawer. Then a sock is randomly picked from the right drawer.

(a) What is the probability that the sock picked from the right drawer is red?
Solution: Let G, R, O denote the events that a green, red, or orange sock is picked from the left drawer respectively. Let E denote the event that a red sock is picked from the right drawer.

$$\begin{aligned} P(E) &= P(G)P(E|G) + P(R)P(E|R) + P(O)P(E|O) \\ &= 0.2 \times 0.4 + 0.3 \times 0.6 + 0.5 \times 0.4 \\ &= 0.08 + 0.18 + 0.2 = 0.46 \end{aligned}$$

(b) What is the probability that the sock picked from the left drawer (and transferred to the right drawer) is red given that the sock picked from the right drawer is red?
Solution:

$$P(R|E) = \frac{P(RE)}{P(E)} = \frac{P(R)P(E|R)}{P(E)} = \frac{0.3 \times 0.6}{0.46} = \frac{9}{23}$$

6. **[12 points]** A bag contains $n \geq 11$ balls labelled 1 through n.

(a) Balls are randomly taken out one at a time.
Obtain $P\{\text{ball number 11 is the } 10^{th} \text{ ball to come out}\}$.

Solution: There are $n!$ equally likely possibilities for the order the balls are drawn out. We are interested only in those orderings that have ball 11 in the 10^{th} place, so we fix ball 11 in that place. There are then $(n-1)!$ possible orderings of the other $n-1$ balls. So

$$P\{\text{ball number 11 is the } 10^{th} \text{ ball to come out}\} = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

ALTERNATIVELY, we can build up a random ordering by selecting one of n places for ball 11, then one of $n-1$ of the remaining places for ball 1, one of $n-2$ places for ball 2, and so on. Thus, any of the n positions for ball 11 are equally likely, so ball 11 is in the 10^{th} place with probability $1/n$.

(b) Suppose that the n balls are randomly assigned to n people that are lined up. The first person in line compares his/her ball with the second person and the one with the smallest numbered ball wins. The winner then compares his/her ball with the third person in line and the one with the smallest numbered ball wins. And so on. Obtain $P\{\text{first person wins exactly ten rounds}\}$.

Solution: This is similar to a quiz problem. If $n > 11$, for the first person to win exactly ten rounds, her number has to be smaller than those of the second through eleventh persons in line, but larger than that of the twelfth person in line. What happens to the remaining people in line is irrelevant. So, there are 12 people with 12 different numbers that we care about. The numbers were randomly assigned, which can occur in 12! ways. We are interested only in those assignments which have the smallest number assigned to the twelfth person and the second smallest assigned to the first person, so we are free to assign the remaining ten numbers to the second through eleventh persons in line, which can be done in 10! ways. Hence $P\{\text{first person wins exactly ten rounds}\} = \frac{10!}{12!} = \frac{1}{132}$. An alternative way to do the final calculation is to let A be the event the 12^{th} person has the smallest ball among the first 12 people, and let B be the event the first person has the second smallest ball among the first 12 people. Then $P(AB) = P(A)P(B|A) = \frac{1}{12}\frac{1}{11} = \frac{1}{132}$.
If $n = 11$, then there is no twelfth person in line, so for the first person to win exactly ten rounds, her number has to be smaller than those of the second through eleventh persons in line, and hence $P\{\text{first person wins exactly ten rounds}\} = \frac{10!}{11!} = \frac{1}{11}$.