**University of Illinois** Spring 2015

# ECE 313: Final Exam

Monday, May 11, 2015
1:30 p.m. — 4:30 p.m.

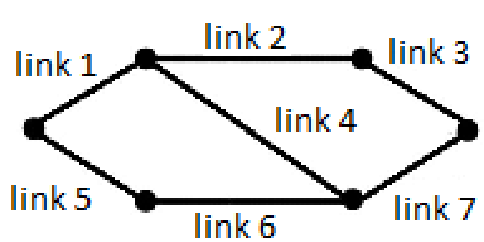
1. **[18 points]** You are in a gambling game, which has four overlapping events: A, B, C and D. For all parts of this problem, $P(A) = \frac{1}{2}$.

   (a) $P(AB) = \frac{1}{5}$. Events $A$ and $B$ are independent. What is $P(A^cB^c)$?
   **Solution:**
   $$P(B) = \frac{P(AB)}{P(A)} = \frac{2}{5}.$$
   Since events $A$ and $B$ are independent,
   $$P(A^cB^c) = P(A^c)P(B^c) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{5}\right) = \frac{3}{10}.$$

   (b) $P(C) = \frac{1}{2}$, and $P(AC) = \frac{1}{6}$. What is $P(A^cC^c)$?
   **Solution:**
   $$P(A^cC) = P(C) - P(AC) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.$$
   $$P(A^cC^c) = P(A^c) - P(A^cC) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

   (c) $P(A|D) = \frac{2}{3}$. What is the maximum possible values of $P(D)$?
   **Solution:**
   $$P(D) = \frac{P(AD)}{P(A|D)} \leq \frac{P(A)}{P(A|D)} = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4}.$$

2. **[20 points]** Consider the following s-t network, where link $i \in \{2, \cdots, 6\}$ fails with probability $\frac{1}{2}$, independent of one another, and links 1 and 7 never fail. Let $c_i$ be the capacity of link $i$, with $c_1 = 30$, $c_2 = 10$, $c_3 = 15$, $c_4 = 20$, $c_5 = 20$, $c_6 = 15$, $c_7 = 15$.




   (a) What values can the capacity of this network take?
   **Solution:** It can take values 0, 10, 15, 25.

(b) What is the outage probability?
**Solution:** Let $F$ denote the event of network outage, and $L_i$ denote the event that link $i$ is working. Since links 1 and 7 never fail, $P(F|L_4) = 0$.
$P(F|L_4^c) = P((L_2^c \cup L_3^c)(L_5^c \cup L_6^c)) = (\frac{1}{2} + \frac{1}{2} - (\frac{1}{2})^2)^2 = \frac{9}{16}$.
Using total probability, $P(F) = P(F|L_4)P(L_4) + P(F|L_4^c)P(L_4^c) = 0 + \frac{9}{16}\frac{1}{2} = \frac{9}{32}$

3. **[10 points]** Consider the random variables $(X, Y)$ such that

$$(X,Y) = \begin{cases} (1,4) & \text{with probability } 1/3; \\ (2,2) & \text{with probability } 1/3; \\ (4,1) & \text{with probability } 1/3. \end{cases}$$

Find the value of CDF $F_{X,Y}$ at points

(a) $(2.5, 0.5)$
**Solution:** 0.

(b) $(2.5, 2.5)$
**Solution:** 1/3.

(c) $(1.5, 5.5)$
**Solution:** 1/3.

(d) $(3.5, 4.5)$
**Solution:** 2/3.

4. **[15 points]** The pdf for the r.v. $(X, Y)$ is given by

$$f_{X,Y}(x,y) = \begin{cases} 1/8 & \text{if } \lfloor x \rfloor + \lfloor y \rfloor \text{ is even and } -2 \le x \le 2; -2 \le y \le 2, \\ 0 & \text{otherwise} \end{cases}$$

(here $\lfloor x \rfloor$ is the floor function, the largest integer $\le x$).

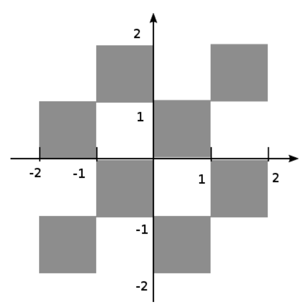



Figure 1: Probability density function $f_{X,Y}$.

(a) Find the marginal pdf for $X$.
**Solution:** This pdf is, obviously, proportional to the lengths of the intersections of the lines $x = \text{const}$ with the support of $f_{X,Y}$. It is 2 on $x \in [-2, 2]$ and 0 otherwise. So, $f_X = 1/4$ on $[-2, 2]$, and 0 elsewhere.

(b) Find $\mathbb{P}(|X-Y| \leq 1)$.

**Solution:** $\mathbb{P}(|X-Y| \leq 1)$ is the total area /8 of the support of the density intersected with $\{|x-y| \leq 1\}$. By inspection, it is 1/2.

5. **[24 points]** Alice, Bob and Charlie are tossing fair coins at the same time, but independently of each other. Let $N$ be the number of these groups of simultaneous tosses until the first time *at least* one of them gets a HEAD on their toss.

(a) Find the probability that $N$ is strictly larger than 2.

**Solution:** The only way $N$ is larger than 2 is if each of Alice, Bob and Charlie land TAILS on each of their first two tosses This is an event of probability $\frac{1}{64}$.

(b) Calculate $E[N]$.

**Solution:** Observe that $P(N > k) = \frac{1}{8}^k$, since the event $N > k$ occurs exactly when each of Alice, Bob and Charlie land TAILS on each of their first $k$ tosses. In other words, $N$ is a geometric random variable with parameter $\frac{7}{8}$. So the mean of $N$ is $\frac{8}{7}$.

(c) Find $E[N^2]$.

**Solution:** Since $N$ is a geometric random variable with parameter $\frac{7}{8}$, we know that its variance is $\frac{8}{49}$. We conclude that $E[N^2] = \text{Var}(N^2) + \mu_N^2 = \frac{8}{49} + \frac{64}{49} = \frac{72}{49}$.

(d) Calculate $E[N^2|N > 2]$.

**Solution:** Since $N$ is a geometric random variable, we can use the memoryless property to argue that conditioned on $N > 2$ we can consider $N$ to be equal to the sum of 2 and a geometric random variable (say $M$) with parameter $\frac{7}{8}$. Then we can write $E[N^2|N > 2] = E[(2+M)^2] = 4 + 2E[M] + E[M^2] = 4 + \frac{16}{7} + \frac{72}{49} = \frac{380}{49}$.

6. **[21 points]** $X$ and $Y$ are two random variables with the following properties: $E[X] = 1$, $E[Y] = 3$, $\text{Var}(X) = 4, \text{Var}(Y) = 1, \rho_{X,Y} = 0.2$.

(a) If $Z = 2X + 3Y - 1$ then find $E[Z]$ and $\text{Var}(Z)$.

**Solution:** Expectation is linear and $E[Z] = 2 + 9 - 1 = 10$. On the other hand, $\text{Var}(Z) = \text{Var}(2X + 3Y) = 4\text{Var}(X) + 9\text{Var}(Y) + 12\text{Cov}(X, Y) = 16 + 9 + 4.8 = 29.8 = \frac{149}{5}$. This is because $\text{Cov}(X, Y) = \rho_{X,Y}\sigma_X\sigma_Y = 0.4$.

(b) Find the optimal *linear* estimate of $X$ given that $Y = 1$.

**Solution:** The optimal linear estimate of $X$ given $Y = 1$ is $\mu_X + \frac{\sigma_X}{\sigma_Y}\rho_{X,Y}(1 - \mu_Y)$ which works out to $0.2 = \frac{1}{5}$.

(c) Suppose $W = X + \alpha Y$. Find the value of $\alpha$ such that $W$ and $X$ are uncorrelated.

**Solution:** $W$ and $X$ are uncorrelated if $E[WX] = \mu_W\mu_X$. Now we have by linearity of expectation that $\mu_X\mu_W = (1 + 3\alpha)$. On the other hand, $E[WX] = E[X^2] + \alpha E[XY] = 5 + 3.4\alpha$. Equating the two terms we get $\alpha = -10$.

7. **[24 points]** Let $X$ and $Y$ be jointly Gaussian random variables. It is known that $\mu_Y = -\frac{2}{9}$ and $\sigma_Y^2 = 4$. It is also known that $\mu_{Y|X} = -\frac{1}{3}X$ and that the resulting minimum mean squared error for the best linear estimator is $MMSE_{\hat{E}[Y|X]} = 3$.

(a) Obtain the conditional pdf $f_{Y|X}(v|u)$ for all $v$ and $u$.

**Solution:** It is known that if $X$ and $Y$ are jointly Gaussian, then given $X = u$, $Y \sim N\left(\mu_{Y|X}, MMSE_{\hat{E}[Y|X]}\right)$. Therefore, $f_{Y|X}(v|u) = \frac{1}{\sqrt{6\pi}} e^{-\frac{1}{2}\left(\frac{3v+u}{3\sqrt{3}}\right)^2}$ for all $-\infty < u, v < \infty$.

(b) Obtain the best unconstrained MSE estimator of $Y$ from $X$.

**Solution:** It is known that the best unconstrained MSE estimator is the conditional mean so $g^*(X) = E[Y|X] = \mu_{Y|X} = -\frac{1}{3}X$.

(c) Obtain $\hat{E}[Y|X = 3]$.

**Solution:** It is known that if $X$ and $Y$ are jointly Gaussian, then the best linear MSE estimator of $Y$ from $X$ is the same as the best unconstrained estimator so $\hat{E}[Y|X = 3] = g^*(3) = -\frac{1}{3}3 = -1$.

(d) Obtain $\mu_X$ and $\sigma_X^2$.

**Solution:** It is known that if $X$ and $Y$ are jointly Gaussian, then marginal distribution of $X$ is Gaussian, so we need to find its mean and variance.
To obtain $\mu_X$ we use the fact that, for jointly Gaussians, the best linear MSE estimator of $Y$ from $X$ is the same as the best unconstrained estimator, which is the conditional mean. So,

$$\begin{aligned} \hat{E}[Y|X] &= E[Y|X] \\ \frac{Cov(X,Y)}{\sigma_X^2}(X - \mu_X) + \mu_Y &= \mu_{Y|X} \\ \frac{Cov(X,Y)}{\sigma_X^2}X + \left(-\frac{Cov(X,Y)}{\sigma_X^2}\mu_X + \mu_Y\right) &= -\frac{1}{3}X \end{aligned}.$$

We can extract the slope of the line, as a function of $X$, as $\frac{Cov(X,Y)}{\sigma_X^2} = -\frac{1}{3}$. We can also see that the $y$-intercept of the line is zero, so that $0 = -\frac{Cov(X,Y)}{\sigma_X^2}\mu_X + \mu_Y$. We can then solve for $\mu_X = \mu_Y \frac{\sigma_X^2}{Cov(X,Y)} = \frac{-2}{9}(-3) = \frac{2}{3}$.
To obtain $\sigma_X^2$, we are told that the MMSE from the linear estimator is $MMSE_{\hat{E}[Y|X]} = 3$ so that

$$\begin{aligned} 3 &= MMSE_{\hat{E}[Y|X]} = \sigma_Y^2 - \frac{Cov^2(X,Y)}{\sigma_X^2} = \sigma_Y^2 - Cov(X,Y)\frac{Cov(X,Y)}{\sigma_X^2} \\ &= 4 - Cov(X,Y)\left(\frac{-1}{3}\right) \end{aligned}.$$

We can solve for $Cov(X,Y) = -3$, and using $\frac{Cov(X,Y)}{\sigma_X^2} = -\frac{1}{3}$ we obtain $\sigma_X^2 = 9$.

8. **[20 points]** Let $N_t$ be a Poisson process with rate 2.

(a) Let $0 < s < t$, obtain $E[N_t N_{t-s}]$.

**Solution:** Notice that the intervals $(0, t)$ and $(0, t-s)$ overlap but the intervals $(0, t-s)$ and $(t-s, t)$ do not. Let $N_t = N_{t-s} + (N_t - N_{t-s})$, then, the random variables $N_t - N_{t-s}$ and $N_{t-s}$ are independent (because their time intervals are non-overlapping. Also notice that $N_t - N_{t-s} \sim Poisson\,(2s)$ and $N_{t-s} \sim Poisson\,(2(t-s))$. Therefore,

$$\begin{aligned} E[N_t N_{t-s}] &= E\left[(N_{t-s} + (N_t - N_{t-s}))\, N_{t-s}\right] = E\left[N_{t-s}^2 + (N_t - N_{t-s})\, N_{t-s}\right] \\ &= E\left[N_{t-s}^2\right] + E\left[N_t - N_{t-s}\right] E\left[N_{t-s}\right] = 2(t-s) + (2(t-s))^2 + 4(t-s)s \end{aligned}$$

(b) Obtain $P\{N_1 = 2 \text{ and } N_3 = 5\}$.

**Solution:** Notice that the intervals $(0,1)$ and $(0,3)$ overlap but the intervals $(0,1)$ and $(1,3)$ do not. Let $N_3 = N_1 + (N_3 - N_1)$, then, the random variables $N_3 - N_1$ and $N_1$ are independent (because their time intervals are non-overlapping. Also notice that $N_3 - N_1 \sim \textit{Poisson}\,(2(3-1))$ and $N_1 \sim \textit{Poisson}\,(2(1))$. Therefore,

$$\begin{aligned} P\{N_1 = 2, N_3 = 5\} &= P\{N_1 = 2, (N_3 - N_1) = 3\} = P\{N_1 = 2\}\, P\{N_3 - N_1 = 3\} \\ &= \tfrac{e^{-2}(2)^2}{2!}\tfrac{e^{-4}(4)^3}{3!} = \tfrac{64}{3}e^{-6} \end{aligned}$$

9. **[18 points]** Recall that if $X$ is a random variable with mean $\mu$ and variance $\sigma^2$, then $Y = \frac{X-\mu}{\sigma}$ defines the standardized version of $X$. For each of the following two choices of distribution for $X$, sketch and carefully label the pmfs of $X$ and $Y$.

(a) $X$ is the number generated by rolling a fair die. Carefully sketch the pmf of $X$ and the pmf of $Y$. (Hint: The standard deviation of $X$ is given by $\sigma \approx 1.7$.)

**Solution:** The mean of $X$ is 3.5, so to get the pmf of $Y$ we shift the pmf of $X$ to the left by 3.5 (i.e. we center it) and then scale by shrinking the pmf horizontally by the factor 1.7. The values of the pmf for $Y$ are still all 1/6.

(b) $X$ has the binomial distribution with parameters $n = 4$ and $p = 0.5$. Carefully sketch the pmf of $X$ and the pmf of $Y$.

**Solution:** The mean of $X$ is $np = 2$ and the variance is $np(1-p) = 1$. Since the variance is already one, $Y = X - 2$; no scaling is necessary. The pmf of $Y$ is the pmf of $X$ shifted to the left by two (i.e. it is centered).

10. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Suppose $X$ and $Y$ are independent, continuous-type random variables. $W = \max(X,Y)$, $Z = \min(X,Y)$.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $F_W(t) = F_X(t)F_Y(t)$ |
| ☐ | ☐ | $F_Z(t) = (1 - F_X(t))(1 - F_Y(t))$ |
| ☐ | ☐ | $f_Z(t) = f_X(t)(1 - F_Y(t)) + f_Y(t)(1 - F_X(t))$ |

**Solution:** True, False, True

(b) Let $A$, $B$, $C$ be independent events with $0 < P(A), P(B), P(C) < 1$.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $P(AC\|B) = P(AC\|B^c)$ |
| ☐ | ☐ | $P(AB\|C) \leq P(C\|AB)P(AB)$ |

**Solution:** True, False

(c) Let $X, Y$ be two independent normal random variables with $\mu_X = \mu_Y$ and with $\sigma_X > \sigma_Y$.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $\mathbb{P}(Y \leq X) > 1/2$. |
| ☐ | ☐ | $\mathbb{P}(X = Y) > 0$ |

**Solution:** False, False

(d) $X$ and $Y$ are jointly distributed discrete random variables. They are uncorrelated if:

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $\mathrm{Var}(X + Y) = \mathrm{Var}(X - Y)$ |
| ☐ | ☐ | $\mathrm{E[XY]} = 0$ |
| ☐ | ☐ | $P(X = u, Y = v) = P(X = u)P(Y = v)$ for every pair $(u, v)$ |

**Solution:** True, False, True