

ECE 313: Hour Exam I

Wednesday, March 4, 2015

7:00 p.m. — 8:15 p.m.

MSEB 100, ECEB 1002, ECEB 2017

1. [20 points] The two parts of this problem are unrelated.
- (a) Suppose that you have five distinct bowls, each can hold up to two oranges. If you have four indistinguishable oranges, how many ways are there to place the oranges in the bowls?
- Solution:** We can partition the problem into three mutually exclusive scenarios. There can be four bowls with one orange each, and there are $\binom{5}{4} = 5$ ways to choose which four out of five bowls have oranges. There can be two bowls with two oranges each. There are $\binom{5}{2} = 10$ ways to choose which two out of five bowls have oranges. There can also be three bowls with oranges: one with two oranges, and two with one orange each. There are $\binom{5}{1} = 5$ ways to choose the bowl to place the two oranges together, and for each one of those choices there are $\binom{4}{2} = 6$ ways to choose the two bowls to place the other two oranges with one orange per bowl. So, there are $5(6) = 30$ ways for this scenario to occur. This gives a total of $5 + 10 + 30 = 45$ ways.
- (b) Suppose that five men and five women are invited to have dinner at a round table. How many seating arrangements are there if there must not be two men nor two women seating next to each other? (NOTE: a seating plan at a round table is invariant with respect to rotation.)
- Solution:** Fix one man as a reference starting point, then the remaining four men can be seated in $4! = 24$ ways with an empty chair between each pair of neighboring men. Then, for each one of these choices, the five women can be seated in $5! = 120$ ways at the empty chairs. Therefore, there are $24(120) = 2880$ seating arrangements.
2. [15 points] Two random bits (equally likely to be 0 or 1) are drawn independently from a file. Let X denote the sum of the two bits.
- (a) Find $E[X]$
- Solution:** The pmf of X is the following: $P(X = 0) = P(X = 2) = 0.25$ and $P(X = 1) = 0.5$. So $E[X] = 1$.
- (b) Find $\text{Var}(X)$.
- Solution:** We have $E[X^2] = 0.5 + 1 = 1.5$. So $\text{Var}(X) = 1.5 - 1 = 0.5$.
- (c) Find $\text{Var}(3X + 2)$.
- Solution:** $\text{Var}(3X + 2) = \text{Var}(3X) = 9\text{Var}(X) = 4.5$.
- (d) Find $E[\frac{1}{1+X}]$.
- Solution:** Using LOTUS we have $E[\frac{1}{1+X}] = (1 \cdot 0.25) + (\frac{1}{2} \cdot 0.5) + (\frac{1}{3} \cdot 0.25) = \frac{7}{12}$.

3. [15 points] A fair die is rolled. Let X be the number of rolls required until 2 or 3 show for the first time.

(a) Find $E[X]$.

Solution: Since X is a geometric random variable with parameter $p = 2/6 = 1/3$,

$$E[X] = \frac{1}{p} = \frac{1}{1/3} = 3.$$

(b) Find $P(X = 5 \mid X > 3)$.

Solution: By the memoryless property of Geometric random variables,

$$P(X = 5 \mid X > 3) = P(X = 2) = (1 - p)p = \frac{2}{9}.$$

(c) Find $E[X \mid X > 3]$.

Solution: By the memoryless property of Geometric random variables,

$$E[X \mid X > 3] = 3 + E[X - 3 \mid X > 3] = 3 + \frac{1}{p} = 6.$$

4. [15 points] A hundred cars participate in a race. The probability that a car breaks down and drops out of the race is 0.01.

(a) Find the probability that all cars finish the race.

Solution:

$$P(\text{all cars finish}) = (1 - 0.01)^{100} = 0.99^{100}.$$

(b) Use Poisson approximation to find the probability that exactly one car breaks down. Leave your answer in terms of e .

Solution: The number of cars that break down is a binomial random variable with $n = 100$ and $p = 0.01$. Let X denote the Poisson approximation with

$$\lambda = np = 100 \times 0.01 = 1.$$

$$P(X = 1) = \frac{e^{-1} \times 1}{1!} = e^{-1}.$$

5. [20 points] Peter has a biased coin, which has $P\{\text{heads}\} = 1/3$. He flips the coin twice. Let X be the number of heads that show. After Peter flips the coin twice, he rolls a fair die X times. Let Y be the number of times that 3 shows.

- (a) Compute the probability that $Y = 1$ (i.e., compute $P\{Y = 1\}$).

Solution: Using the law of total probability

$$P(Y = 1) = P(Y = 1 | X = 0)P(X = 0) + P(Y = 1 | X = 1)P(X = 1) + P(Y = 1 | X = 2)P(X = 2)$$

Notice that X is a $Binomial(2, 1/3)$ random variable, so $P\{X = k\} = \binom{2}{k}(1/3)^k(2/3)^{2-k}$ for $k \in \{0, 1, 2\}$.

$P(Y = 1 | X = 0) = 0$, because Peter cannot roll a 3 if he didn't roll the die at all.

$P(Y = 1 | X = 1) = 1/6$, because Peter only rolled the die once, and there is a 1/6 chance of rolling a 3.

$P(Y = 1 | X = 2) = \binom{2}{1}(1/6)(5/6) = 5/18$, because Peter rolled the die twice, and there is a 1/6 chance of rolling a 3 in each roll.

Therefore,

$$P(Y = 1) = 0 + (1/6)\binom{2}{1}(1/3)^1(2/3)^{2-1} + (5/18)\binom{2}{2}(1/3)^2(2/3)^{2-2} = 17/162$$

- (b) Suppose Peter rolled 3 once. Compute the probability that X was 2 (i.e., compute $P\{X = 2 | Y = 1\}$).

Solution: Using Bayes rule and the results from part (a)

$$P\{X = 2 | Y = 1\} = \frac{P\{Y = 1 | X = 2\}P\{X = 2\}}{P\{Y = 1\}} = \frac{(5/18)(1/9)}{17/162} = 5/17$$

6. [15 points] A standard deck of cards contains 52 cards (A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K of each suit), while a piquet deck contains only 32 cards (A, 7, 8, 9, 10, J, Q, K of each suit).

- (a) If you are given a card drawn at random from an unknown deck, formulate the ML decision rule whether the deck is the standard one (\mathbf{H}_0) or the piquet deck (\mathbf{H}_1).

Solution: The probability to pick any given value in the full deck is 1/52, which is less than the probability 1/8 to pick a given value in the piquet deck. Hence, if the picked value is one of A, 7, 8, 9, 10, J, Q, K, the ML rule indicates that the deck is piquet one; otherwise (with certainty!) it's a full deck.

- (b) Suppose that the chances to see a piquet deck in the US are one in three. Formulate the corresponding MAP decision rule.

Solution: As the ratio of the conditional probabilities $(1/8)/(1/52) = 13/8 = 1.625$ is less than $\pi_0/\pi_1 > (2/3)/(1/3) = 2$ (i.e. the ratio of chances to see a regular deck over a piquet deck), one should *always* choose the *regular deck*, according to MAP (i.e. no experiments necessary).