

## ECE 313: Final Exam

Monday, May 12, 2014, 7:00 p.m. — 10:00 p.m.

Room 151 Everitt Lab (Sections E/9 and C/10) and 100 MSEB (Sections D/11 and F/1)

1. (a) The distribution of  $X_k$  is Bernoulli with  $p = 0.5$ , so  $p_{x_k}(0) = p_{x_k}(1) = 0.5$ .
- (b) The sequence  $X_1, X_2, \dots$  is a *Bernoulli process*.
- (c) In the first ten minutes, there will be  $(6 \text{ tosses/min}) \times (10 \text{ min}) = 60$  tosses. Let  $C_{60}$  denote the number of heads that appear in 60 tosses; then  $C_{60}$  is binomially distributed with pmf  $p_{C_{60}}(k) = \binom{60}{k}(0.5)^k(0.5)^{60-k} = \binom{60}{k}(0.5)^{60}$ ,  $0 \leq k \leq 60$ .
- (d) Since the tosses are conducted every 10 s, the time elapsed until a heads shows up for the first time is  $T_1 = 10 \times L_1$  [s], where  $L_1$  is a geometric random variable with parameter  $p = 0.5$ . Thus,  $p_{T_1}(10k) = (0.5)^k$ ,  $k = 1, 2, \dots$

2. (a) Since  $T_1$  and  $T_2$  are independent, we can use the convolution formula; thus,  $f_{T_t}(t) = \int_0^t \lambda^2 e^{-\lambda u} e^{-\lambda(t-u)} du = \int_0^t \lambda^2 e^{-\lambda t} du = \lambda^2 t e^{-\lambda t}$   
ALTERNATIVELY, the sum of  $r$  independent exponentially distributed random variables with parameter  $r$  has the Erlang distribution with parameters  $r$  and  $\lambda$ . The Erlang distribution for  $r = 2$  has pdf  $\lambda^2 t e^{-\lambda t}$  for  $t \geq 0$ .
- (b) First we will find the CDF of  $T_s$  and then we will differentiate to obtain the pdf. From the definition of CDF, we have that

$$\begin{aligned}
 F_{T_s}(t) &= P\{T_s \leq t\} = P\{\min\{T_1, T_2\} \leq t\} = 1 - P\{\min\{T_1, T_2\} > t\} \\
 &= 1 - P(\{T_1 > t\} \cap \{T_2 > t\}) \\
 &= 1 - P\{T_1 > t\}P\{T_2 > t\} \quad (\text{by independence of } T_1 \text{ and } T_2) \\
 &= 1 - e^{-\lambda t} e^{-\lambda t} = 1 - e^{-2\lambda t}
 \end{aligned}$$

Differentiating yields  $f_{T_s}(t) = 2\lambda e^{-2\lambda t}$  for  $t \geq 0$ . That is,  $T_s$  has the exponential distribution with parameter  $2\lambda$ .

- (c) Similarly to part (b) above, first we will find the CDF of  $T_p$  and then will differentiate to find its pdf. Again, from the definition of CDF, we have that  $F_{T_p}(t) = P(T_p \leq t) = P(\max\{T_1, T_2\} \leq t) = P(\{T_1 \leq t\} \cap \{T_2 \leq t\})$ . Then, from independence, we have that  $P(\{T_1 \leq t\} \cap \{T_2 \leq t\}) = P(T_1 \leq t)P(T_2 \leq t) = (1 - e^{-\lambda t})(1 - e^{-\lambda t}) = 1 - 2e^{-\lambda t} + e^{-2\lambda t}$ . Finally, by differentiating, we obtain that  $f_{T_p}(t) = 2\lambda(e^{-\lambda t} - e^{-2\lambda t})$ .
  - (d) Let  $D = T_p - T_s$ , so that  $T_p = T_s + D$ . Then  $P\{D > 0\} = 1$ , because the minimum of two numbers is less than the maximum. So  $E[D] > 0$ . Therefore,  $E[T_p] = E[T_s] + E[D] > E[T_s]$ .
3. (a)  $P\{WWW\} = P\{LLL\} = 1/3 \times ((1/4)^3 + (3/4)^3 + (1/2)^3) = 3/16$ . So the probability the match ends in three sets is  $P\{WWW, LLL\} = 3/8$ .
  - (b)  $1/2$  by symmetry.
  - (c)

$$\begin{aligned}
 P(\text{lost 1-3}) &= P\{WLLL, LWLL, LLWL\} = 3P\{WLLL\} \\
 &= 3 \cdot \frac{1}{3} \cdot ((1/4)^3(3/4) + (3/4)^3(1/4) + (1/2)^4) \\
 &= \frac{3}{256} + \frac{27}{256} + \frac{16}{256} = \frac{46}{256} = \frac{23}{128}
 \end{aligned}$$

(d)  $\frac{3}{3+27+16} = \frac{3}{46}$ .

4. (a) No, the support of  $f_{X,Y}$  is not a product set.  
 (b) Conditioned on  $X = u$ ,  $Y$  is uniformly distributed on  $[0, u]$ . Therefore  $g^*(u) = E[Y|X = u] = u/2$ . The MMSE is  $E[(Y - g^*(X))^2] = E[Y^2] - E[g^*(X)^2]$ , where  $E[Y^2] = \int_0^1 2(1-v)v^2 dv = \frac{1}{6}$ ,  $E[X^2] = \int_0^1 2uu^2 du = \frac{1}{2}$ . So

$$\text{MMSE} = E[Y^2] - E[X^2]/4 = \frac{1}{24}.$$

- (c) Since  $g^*$  is linear, it is also the best linear estimator. Therefore  $L^*$  coincides with  $g^*$  and the MSE is also  $\frac{1}{24}$ . Alternatively, we could use the formulas for  $L^*$  and for the associated MSE.  
 (d) By symmetry,  $\sigma_X = \sigma_Y$ . And by the general formula,  $L^*(X) = \mu_Y + \frac{\rho_{X,Y}\sigma_Y}{\sigma_X}(X - \mu_X)$ , whereas  $L^*(X) = \frac{X}{2}$ . Matching the slopes of these two expressions for  $L^*$  yields  $\rho_{X,Y} = 1/2$ .  
 ALTERNATIVELY, we can calculate  $\text{Cov}(X, Y)$ ,  $\text{Var}(X)$ , and  $\text{Var}(Y)$  (which is equal to  $\text{Var}(X)$ ) and use the definition of  $\rho_{X,Y}$ .
5. (a)  $P\{X \geq 3Y + 1\} = P\{X - 3Y \geq 1\}$ . Since  $X$  and  $Y$  are jointly Gaussian,  $X - 3Y$  is a Gaussian random variable. Moreover, it has mean zero and  $\text{Var}(X - 3Y) = \text{Var}(X) - 6\text{Cov}(X, Y) + 9\text{Var}(Y) = 1 - 6\rho + 9 = 10 - 6\rho$ . Thus,

$$P\{X \geq 3Y + 1\} = P\left\{\frac{X - 3Y}{\sqrt{10 - 6\rho}} \geq \frac{1}{\sqrt{10 - 6\rho}}\right\} = Q\left(\frac{1}{\sqrt{10 - 6\rho}}\right)$$

- (b) By the formula for  $\hat{E}[Y|X]$  and the associated mean square error, we have  $\hat{E}[Y|X = 2] = 2\rho$ , and the mean square error for estimating  $Y$  from  $X$  is  $\sigma_e^2 = 1 - \rho^2$ . Thus, the conditional distribution of  $Y$  given  $X = 2$  is the Gaussian distribution with mean  $2\rho$  and variance  $1 - \rho^2$ . Equivalently, the  $N(2\rho, 1 - \rho^2)$  distribution, or

$$f_{Y|X}(v|2) = \frac{1}{\sqrt{2\pi(1 - \rho^2)}} \exp\left(-\frac{(v - 2\rho)^2}{2(1 - \rho^2)}\right)$$

- (c) For any value of  $\alpha$ , the random variables  $X + \alpha Y$  and  $Y$  are jointly Gaussian, so they are independent if and only if they are uncorrelated. That is, if and only if  $0 = \text{Cov}(X + \alpha Y, Y)$ . Equivalently, if and only if  $0 = \rho + \alpha$ , or  $\alpha = -\rho$ .
6. (a) Let  $X$  denote the number of passengers that arrive for the flight. By the assumptions,  $X$  has the binomial distribution with parameters  $n = 192$  and  $p = 0.75$ . Thus  $E[X] = (192)(0.75) = 144$  and  $\text{Var}(X) = 192(0.75)(0.25) = \frac{3 \cdot 192}{16} = 36$ . So using the Gaussian approximation without a continuity correction,

$$P\{X > 150\} = P\left\{\frac{X - 144}{6} > \frac{150 - 144}{6}\right\} \approx Q\left(\frac{150 - 144}{6}\right) = Q(1).$$

For the continuity correction we would start with  $P\{X \geq 150.5\}$  yielding  $Q(6.5/6)$ . We could also start with  $P\{X \geq 151\}$ , giving  $Q(7/6)$ .

- (b) If we let  $Y_i$  denote the number of passengers that arrive from the  $i^{\text{th}}$  pair, then the total number of passengers that arrive can be written as  $S = Y_1 + \dots + Y_{96}$ , where the  $Y$ 's are independent and  $P\{Y_i = 2\} = p$  and  $P\{Y_i = 0\} = 1 - p$ . Thus,  $S$  has mean  $96(2)p = 144$

and variance  $96(4)p(1-p) = 72$ . (So the effect of pairing gives a larger variance than in part (a).) Thus,

$$P\{S > 150\} = P\left\{\frac{S - 144}{\sqrt{72}} > \frac{150 - 144}{\sqrt{72}}\right\} \approx Q\left(\frac{150 - 144}{\sqrt{72}}\right) = Q(1/\sqrt{2}).$$

(Slightly different correct answers are possible as for part (a). Another way to solve this part is to focus on the number of pairs of passengers that arrive, with the flight being oversold if more than 75 pairs of passengers arrive.)

7.

$$f_0(u) = \begin{cases} \frac{1}{2} & |u| \leq 1 \\ 0 & |u| \geq 1 \end{cases} \quad f_1(u) = \begin{cases} 1 - |u| & |u| \leq 1 \\ 0 & |u| \geq 1 \end{cases}$$

(a) ML decision rule: declare  $H_1$  if  $|X| < \frac{1}{2}$ , and declare  $H_0$  otherwise.

(b)

$$p_{\text{false alarm}} = P(\text{declare } H_1 | H_0 \text{ is true}) = P\left(-\frac{1}{2} \leq X \leq \frac{1}{2} \middle| H_0\right) = \frac{1}{2}.$$

$$p_{\text{miss}} = P(\text{declare } H_0 | H_1 \text{ is true}) = P\left(|X| \geq \frac{1}{2} \middle| H_1\right) = \frac{1}{4}.$$

(c) In order for  $H_0$  to be declared with probability one, we need  $\frac{f_1(u)}{f_0(u)} \leq \frac{\pi_0}{\pi_1}$  for all  $u$ . The maximum of  $\frac{f_1(u)}{f_0(u)}$  is 2 (occurs at  $u = 0$ —draw a sketch) so we need  $2 \leq \frac{\pi_0}{\pi_1}$ , or  $\pi_0 \geq \frac{2}{3}$ .

(d)

$$P\left(H_0 \middle| |X| < 0.5\right) = \frac{P(H_0, |X| < 0.5)}{P(|X| < 0.5)} = \frac{0.2 \times 0.5}{0.2 \times 0.5 + 0.8 \times 0.75} = \frac{1}{7}$$

8. (a)

$$E[X^2] = \int_0^2 0.5u^3 du = \frac{u^4}{8} \Big|_0^2 = 2.$$

(b)

$$P(\lfloor X^2 \rfloor = 1) = P(1 \leq X^2 < 2) = P(1 \leq X < \sqrt{2}) = \int_1^{\sqrt{2}} 0.5u du = \frac{u^2}{4} \Big|_1^{\sqrt{2}} = \frac{1}{4}.$$

(c) The support of  $Y$  is  $(-\infty, \ln 2]$ .

$$F_Y(c) = P\{Y \leq c\} = P\{\ln X \leq c\} = P\{X \leq e^c\} = \int_0^{e^c} 0.5u du = \frac{u^2}{4} \Big|_0^{e^c} = \frac{e^{2c}}{4}$$

for  $c \leq \ln 2$ , and  $F_Y(c) = 1$  for  $c > \ln 2$

(d)

$$F_X(c) = P\{X \leq c\} = \int_0^c 0.5u du = \frac{u^2}{4} \Big|_0^c = \frac{c^2}{4}.$$

Let  $\frac{c^2}{4} = u$ , then  $c = 2\sqrt{u}$ , hence  $g(u) = F^{-1}(u) = 2\sqrt{u}$ , for  $0 \leq u \leq 1$ .

9. (a) True, False

(b) True, False.

(c) True, False.

(d) False, True.

(e) False, True