

ECE 313: Hour Exam II

Monday, April 14, 2014

7:00 p.m. — 8:15 p.m.

Rooms 141 (Sections E,C, and those in D with last names A-K) and 151 Loomis Lab

1. (a) Let N denote the event of an unwanted nuclear attack, and let F_i denote the event of a failure in switch $i = 1, 2$.
- (i) $P(N) = P(F_1 \cup F_2) = P(F_1) + P(F_2) - P(F_1)P(F_2) = 2p - p^2$
- (ii) $P(N) = P(F_1 \cup F_2) \leq P(F_1) + P(F_2) = 2p$.

(b)

$$P(N) = P(F_1 F_2) = P(F_1)P(F_2) = p^2$$

- (c) Switches in series. The essential point is that the failure probability is much smaller for the switches in series. It might be pointed out there is an inconvenience to having switches in series: both buttons have to be pushed to activate rather than just one, but that difference is trivial.

2. (a) The event $\{X^2 + 1 \leq -203.57\}$ is the empty set, so it has probability zero.

(b)

$$\begin{aligned} P\{X^2 - 2X \leq 0\} &= P\{X(X - 2) \leq 0\} = P\{0 \leq X \leq 2\} \\ &= P\left\{\frac{-\mu}{\sigma} \leq \frac{X - \mu}{\sigma} \leq \frac{2 - \mu}{\sigma}\right\} \\ &= \Phi\left(\frac{2 - \mu}{\sigma}\right) - \Phi\left(\frac{-\mu}{\sigma}\right). \end{aligned}$$

There are other ways to write the solution in terms of Φ and/or Q .

3. (a) $\frac{P\{N_1=1, N_3-N_1=0\}}{P\{N_3=1\}} = \frac{P\{N_1=1\}P\{N_3-N_1=0\}}{P\{N_3=1\}} = \frac{\lambda e^{-\lambda} e^{-2\lambda}}{3\lambda e^{-3\lambda}} = \frac{1}{3}$.

ALTERNATIVELY, though it is not emphasized in the course, it turns out that given the number of arrivals of a Poisson process within an interval of time, the distribution of arrival times within the interval are as if they are independent and uniform. Thus, given one arrival during $[0,3]$, it falls within the first third of the interval with probability one third.

- (b) Zero. The number of arrivals in the length zero interval $[1, 1]$ has the Poisson distribution with parameter $\lambda(1 - 1) = 0$, and hence there are no arrivals in that interval, with probability one. Another way to see this is to let T_k be the arrival time of the k th bus, which is a continuous-type random variables for each $k \geq 1$. Hence the probability any of them takes a given value is zero. Formally,

$$P\{\text{there is a bus arriving at } t = 1\} = P\left(\bigcup_{k=1}^{\infty} \{T_k = 1\}\right) \leq \sum_{k=1}^{\infty} P\{T_k = 1\} = 0.$$

- (c) The number of buses to arrive in the length one interval after Alice arrives has the Poisson distribution with parameter λ . So the probability of no arrivals in that interval, i.e. the probability Alice ends up walking home, is $e^{-\lambda}$.

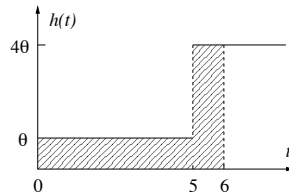
4. (a) $P\{W \leq 2.88\} = P\{2I^2 \leq 2.88\} = P\{I^2 \leq 1.44\} = P\{I \leq 1.2\} = \frac{0.2}{2} = 0.1$.

(b) Using the LOTUS property, $E[W] = E[2I^2] = 2 \int_1^3 u^2 \frac{1}{2} du = \frac{u^3}{3} \Big|_1^3 = \frac{26}{3}$.

(c) The random variable W takes values in the interval $[2, 18]$. For c in that interval, $F_W(c) = P\{2I^2 \leq c\} = P\{I \leq \sqrt{\frac{c}{2}}\} = \frac{1}{2}(\sqrt{\frac{c}{2}} - 1)$. Therefore,

$$F_W(c) = \begin{cases} 0 & \text{for } c \leq 2 \\ \frac{1}{2}(\sqrt{\frac{c}{2}} - 1) & \text{for } 2 \leq c \leq 18 \\ 1 & \text{for } c \geq 18 \end{cases}$$

5. (a) $P\{T \geq 6\} = \exp\left(-\int_0^6 h(t)dt\right) = \exp(-9\theta)$. (Here we used the fact that the integral of h over the interval $[0, 5]$ is 5θ and the integral of h over $[5, 6]$ is 4θ . See the figure.)

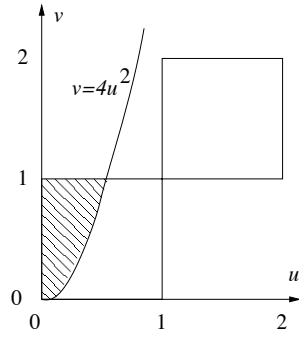


(b) Since $h(s) = \theta$ for $0 \leq s \leq 5$, it follows that $P\{T \leq t\} = 1 - \exp\left(-\int_0^t \theta ds\right) = 1 - e^{-\theta t}$ for $0 \leq t \leq 5$. Therefore, taking the derivative with respect to t , $f_T(t) = \theta e^{-\theta t}$ for $0 \leq t \leq 5$. This is the same as the pdf for an exponentially distributed random variable with rate parameter θ , as one might expect. Therefore, the likelihood of observing $T = 4$ is $f_T(4) = \theta e^{-4\theta}$, and this likelihood is to be maximized. The derivative of the likelihood (with respect to θ) is $(1 - 4\theta)e^{-4\theta}$. It is positive for $\theta < 0.25$ and negative for $\theta > 0.25$. So $\hat{\theta}_{ML} = 0.25$ (This optimization is the same as for estimation of the parameter of an exponentially distributed random variable with parameter θ .)

6. (a) No. The support is not a product set.
 (b) Observe that X has a uniform distribution over $[0, 2]$. Hence, $f_X(u) = 0.5$ for $u \in [0, 2]$ and $f_X(u) = 0$ otherwise.
 (c)

$$f_{X|Y}(u|1.5) = \begin{cases} 1 & 1 \leq u \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(d) The probability is the integral of the pdf over the shaded region



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$$\begin{aligned}
 P\{Y \geq 4X^2\} &= \int_0^{0.5} \int_{4u^2}^1 0.5 dv du \\
 &= \int_0^{0.5} 0.5(1 - 4u^2) du = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}.
 \end{aligned}$$