ECE 313: Problem Set 13: Solutions
Covariance and MMSE Estimation

1. **[Some moments for a random rectangle]**

   (a)

$$\begin{aligned} E[A] &= E[XY] = E[X]E[Y] \quad \text{because } X \text{ and } Y \text{ are independent} \\ &= \left(\frac{1}{2}\right)^2 = \frac{1}{4} \end{aligned}$$

$$\begin{aligned} E[L] &= 2E[X] + 2E[Y] \quad \text{by linearity of expectation} \\ &= 2 \end{aligned}$$

   (b)

$$\begin{aligned} E[A^2] &= E[X^2Y^2] = E[X^2]E[Y^2] \quad \text{because } X^2 \text{ and } Y^2 \text{ are independent} \\ &= \left(\frac{1}{3}\right)^2 = \frac{1}{9} \end{aligned}$$

   Therefore, $\mathrm{Var}(A) = E[A^2] - E[A]^2 = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}$.

   (c)

$$\begin{aligned} \mathrm{Var}(L) &= \mathrm{Var}(2(X+Y)) = 4\mathrm{Var}(X+Y) \quad \text{variance scales quadratically} \\ &= 4(\mathrm{Var}(X) + \mathrm{Var}(Y)) \quad \text{variance of sum is sum of variances for independent r.v.'s} \\ &= 4\left(\frac{1}{12} + \frac{1}{12}\right) = \frac{2}{3} \end{aligned}$$

   (d) $E[AL] = E[2XY(X+Y)] = 2E[X^2]E[Y] + 2E[X]E[Y^2] = 2 \cdot \frac{1}{3} \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{2}{3}$.
   Therefore, $\mathrm{Cov}(A, L) = E[AL] - E[A]E[L] = \frac{2}{3} - \frac{1}{4} \cdot 2 = \frac{1}{6}$.

   (e) $\rho_{A,L} = \frac{\mathrm{Cov}(A,L)}{\sqrt{\mathrm{Var}(A)\mathrm{Var}(L)}} = \frac{\frac{1}{6}}{\sqrt{\frac{7}{144}\frac{2}{3}}} = \sqrt{\frac{6}{7}} \approx 0.92582.$

2. **[Jointly Distributed Random Variables]**
   $E[W] = E[3X + Y + 2] = 3E[X] + E[Y] + 2 = 9.$
   $Var(W) = Var(3X + Y + 2) = 3^2 \cdot Var(X) + Var(Y) + 2 \cdot 3 \cdot 1 \cdot Cov(X, Y) = 9 \cdot 4 + 9 + 6 \cdot 2 \cdot 3 \cdot 0.1$
   $= 48.6.$

3. **[Linear Minimum Mean Square Error Estimation]**
   $\frac{dMSE}{da} = 2E[Y - (a + bX_1 + cX_2)(-1)] = 0 \rightarrow a = \mu_Y - b\mu_{X_1} - c\mu_{X_2}$
   $\frac{dMSE}{db} = 2E[Y - (a + bX_1 + cX_2)(-X1)] = 0 \rightarrow bE[X_1^2] = E[X_1Y] - a\mu_{X_1} - c\mu_{X_1}\mu_{X_2}$
   $b(Var(X_1) + {\mu_{X_1}}^2) = Cov(X_1, Y) + E[X_1]E[Y] - a\mu_{X_1} - c\mu_{X_1}\mu_{X_2}$
   $\frac{dMSE}{dc} = 0 \rightarrow c(Var(X_2) + {\mu_{X_2}}^2) = Cov(X_2, Y) + E[X_2]E[Y] - a\mu_{X_2} - b\mu_{X_1}\mu_{X_2}$
   Plugging in a, $b(Var(X_1) + {\mu_{X_1}}^2) = Cov(X_1, Y) + E[X_1]E[Y] - (\mu_Y - b\mu_{X_1} - c\mu_{X_2})\mu_{X_1} - c\mu_{X_1}\mu_{X_2}$.
   $b(Var(X_1) + {\mu_{X_1}}^2) = Cov(X_1, Y) + b\mu_{X_1}$ and $b = \frac{Cov(X_1,Y)}{Var(X_1)}$. Similarly, $c = \frac{Cov(X_2,Y)}{Var(X_2)}$ and $a =$ $\mu_Y - \frac{Cov(X_1,Y)\mu_{X_1}}{Var(X_1)} - \frac{Cov(X_2,Y)\mu_{X_2}}{Var(X_2)}$

4. **[Covariance I]**

(a)

$$\begin{aligned}\mathrm{Var}(X+2Y) &= \mathrm{Cov}(X+2Y, X+2Y)\\ &= \mathrm{Var}(X)+4\mathrm{Var}(Y)+4\mathrm{Cov}(X,Y)=40\end{aligned}$$

Similarly, $\mathrm{Var}(X-2Y)=\mathrm{Cov}(X-2Y,X-2Y)=\mathrm{Var}(X)+4\mathrm{Var}(Y)-4\mathrm{Cov}(X,Y)=20$. Taking the difference of the two equations describing $\mathrm{Var}(X+2Y)$ and $\mathrm{Var}(X-2Y)$ yields $\mathrm{Cov}(X,Y)=2.5$.

(b) Adding the two equations describing $\mathrm{Var}(X+2Y)$ and $\mathrm{Var}(X-2Y)$, we get

$$\begin{aligned}2\mathrm{Var}(X)+8\mathrm{Var}(Y) &= 60\\ 12\mathrm{Var}(Y)=60\end{aligned}$$

Hence, $\mathrm{Var}(Y)=5$, $\mathrm{Var}(X)=10$, and

$$\rho_{X,Y}=\frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}=0.3536$$

The next two parts are independent of parts (a) and (b), and of each other. In particular, the numbers from part (a) are not to be assumed.

(c) From the expressions for $\mathrm{Var}(X+2Y)$ and $\mathrm{Var}(X-2Y)$ in terms of the variances and covariances in part (a), the condition $\mathrm{Var}(X+2Y)=\mathrm{Var}(X-2Y)$ implies that $\mathrm{Cov}(X,Y)=0$. Hence, X and Y are uncorrelated.

(d) No. The condition $\mathrm{Var}(X)=\mathrm{Var}(Y)$ does not imply that $\mathrm{Cov}(X,Y)=0$.

5. **[Covariance II]**

(a) $\mathrm{Cov}(3X+2,5Y-1)=\mathrm{Cov}(3X,5Y)=15\mathrm{Cov}(X,Y)$.

(b)

$$\begin{aligned}\mathrm{Cov}(2X+1,X+5Y-1) &= \mathrm{Cov}(2X,X+5Y)=\mathrm{Cov}(2X,X)+\mathrm{Cov}(2X,5Y)\\ &= 2\mathrm{Cov}(X,X)+10\mathrm{Cov}(X,Y)=2\mathrm{Var}(X)+10\mathrm{Cov}(X,Y)\end{aligned}$$

(c)

$$\begin{aligned}\mathrm{Cov}(2X+3Z,Y+2Z) &= \mathrm{Cov}(2X,Y)+\mathrm{Cov}(2X+2Z)+\mathrm{Cov}(3Z,Y)+\mathrm{Cov}(3Z,2Z)\\ &= 2\mathrm{Cov}(X,Y)+4\mathrm{Cov}(X,Z)+3\mathrm{Cov}(Z,Y)+6\mathrm{Cov}(Z,Z)\\ &= 2\mathrm{Cov}(X,Y)+6\mathrm{Var}(Z)\end{aligned}$$

6. **[Covariance III]**

(a) In general, for $S=\frac{X_1+X_2}{2}$.

$$\begin{aligned}E[S] &= \mu_S=E\left[\frac{X_1+X_2}{2}\right]=\mu\\ \sigma_S^2 &= \frac{\mathrm{Var}(X_1+X_2)}{4}=\frac{2\sigma^2+2\mathrm{Cov}(X_1,X_2)}{4}=\frac{\sigma^2+\mathrm{Cov}(X_1,X_2)}{2}\\ SNR_S &= \frac{2\mu^2}{\sigma^2+\mathrm{Cov}(X_1,X_2)}\end{aligned}$$

Thus, if X_1 and X_2 are uncorrelated, $SNR_S=\frac{2\mu^2}{\sigma^2}=2SNR_X$. Thus, averaging improves the SNR by a factor equal to the number of observations being averaged, if the observations are uncorrelated.

(b) Since $\mathrm{Cov}(X_1, X_2) = \sigma^2 \rho_{X_1,X_2}$, the formula above for SNR_S is equivalent to

$$SNR_S = \frac{2\mu^2}{\sigma^2(1 + \rho_{X_1 X_2})}.$$

Setting SNR_S equal to $1.5\frac{\mu^2}{\sigma^2}$ yields $\rho_{XY} = \frac{1}{3}$.

(c) $SNR_S \to \infty$ as $\rho_{X_1 X_2} \to -1$.