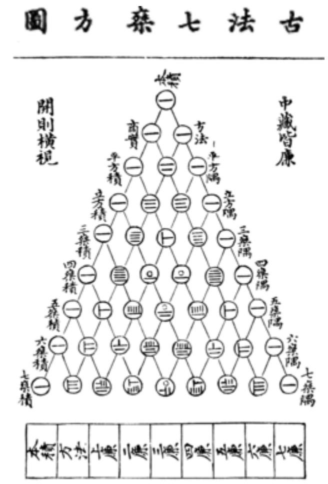
University of Illinois **Spring 2013**

ECE 313: Problem Set 12: Problems and Solutions
Sums of RVs, 1D functions of joint RVs

Due: Wednesday, April 17 at 6 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.5–4.6

1. **[Binomial Random Variables]**
A binomial RV can be written as the sum of n iid (independent and identically distributed) Bernoulli RVs, while a negative binomial RV can be written as the sum of n iid geometric random variables. Both are related, in an unexpected way, to a beautiful diagram called the binomial triangle. The binomial triangle was apparently first described about 2000 years ago by the Sanskrit author Pingala, and has been the subject of papers written by wine-loving mathematician genius poet gamblers including Omar Khayyam, Yang Hui and Blaise Pascal.



The binomial triangle, as drawn by Yang Hui in the 13th century (courtesy of wikipedia).

(a) The binomial triangle is a method for finding the coefficients of the n^{th} power of a second-order polynomial. Suppose that

$$(x+y)^n = \sum_{k=0}^{n} a_n[k] x^{n-k} y^k$$

The binomial triangle shows that the coefficients $a_n[k]$ can be derived in a recursion, as $a_n[k] = a_{n-1}[k-1] + a_{n-1}[k]$. As modern mathematicians, we are particularly interested in the fact that this recursion can be written

$$a_n[k] = a_{n-1}[k] * h[k]$$

What is $h[k]$?
Solution:

$$h[k] = \begin{cases} 1 & 0 \le k \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) A binomial random variable can be written as the sum of n independent and identically distributed Bernoulli random variables. In particular, suppose that

$$p_X(k) = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}, \qquad p_B(k) = \begin{cases} p & k = 1 \\ (1-p) & k = 0 \end{cases}$$

If we define $Y = X + B$, then it is also true that $Y \sim \text{binomial}(n+1, p)$. Prove that the pmf of Y is given by $p_Y(k) = p_X(k) * p_B(k)$.
Solution:

$$\begin{aligned}
p_Y(k) &= p_B(1)p_X(k-1) + p_B(0)p_X(k), \quad 0 \le k \le n+1 \\
&= \left(\begin{array}{c} n \\ k-1 \end{array}\right) p^k(1-p)^{n-k} + \left(\begin{array}{c} n \\ k \end{array}\right) p^k(1-p)^{n-k} \\
&= \left(\frac{1}{(n-k+1)!(k-1)!} + \frac{1}{(n-k)!k!}\right) n! p^k (1-p)^{n-k} \\
&= \left(\frac{k + (n+1-k)}{(n+1-k)!k!}\right) n! p^k (1-p)^{n-k} \\
&= \left(\begin{array}{c} n+1 \\ k \end{array}\right) p^k(1-p)^{n-k}, \quad 0 \le k \le n+1
\end{aligned}$$

2. **[Uniform Gophers and Geometric Gophers]**
In the game Gopherquest, gamers grab golden gophers with globs of gooey grapefruit gum. The game begins by showing an empty field, dotted with gopher holes. After exactly S seconds, a silver gopher appears; this is a signal telling the gamer to prepare. Then, at time $T = S + G$, a golden gopher appears. The random variables in question have the following pmfs:

$$p_S(k) = \begin{cases} p(1-p)^{k-1} & k \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

$$p_G(k) = \begin{cases} \frac{1}{10} & 1 \le k \le 10 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the conditional pmf $p_{T|S}(i|k)$.
Solution: The conditional pmf is just the pmf that $G = T - S$, i.e.,

$$p_{T|S}(i|k) = \begin{cases} \frac{1}{10} & k+1 \le i \le k+10 \\ 0 & \text{otherwise} \end{cases}$$

(b) Find the pmf $p_T(i)$.
Solution:

$$\begin{aligned}
p_T(i) &= p_G(i) * p_S(i) \\
&= \sum_{k=i-10}^{i-1} p_S(k) p_G(i-k) \\
&= \frac{p}{10} \sum_{k=i-10}^{i-1} (1-p)^k \\
&= \frac{p}{10} \sum_{k=i-10}^{\infty} (1-p)^k - \sum_{k=i}^{\infty} (1-p)^k \\
&= \frac{p}{10} \frac{(1-p)^{i-10} - (1-p)^i}{1-(1-p)} \\
&= \frac{(1-p)^i}{10} \left(\frac{1}{(1-p)^{10}} - 1\right)
\end{aligned}$$

3. **[Noisy Signals]**
Consider the following two hypotheses:

$$\begin{aligned} H_0 &: \quad Z = X + N \\ H_1 &: \quad Z = Y + N \end{aligned}$$

where $X \sim \mathcal{U}(0,1)$ (uniform between 0 and 1), $Y \sim \mathcal{U}(1,2)$ (uniform between 1 and 2), $N \sim \mathcal{U}(-b,b)$ (uniform between -b and b, for some constant b). Define $f_0(u)$ to be the pdf of Z under hypothesis H_0, and $f_1(u)$ to be the pdf of Z under hypothesis H_1.

(a) Sketch $f_0(u)$, assuming that $b \leq \frac{1}{2}$. Show the value of both u and $f_0(u)$ at every slope discontinuity.
Solution: The slope discontinuities occur at

$$\begin{aligned} f_0(-b) &= 0 \\ f_0(b) &= 1 \\ f_0(1-b) &= 1 \\ f_0(1+b) &= 0 \end{aligned}$$

(b) Find the maximum likelihood decision rule, and specify its false alarm probability P_{FA} and missed detection probability P_{MD}, on the assumption that $b \leq \frac{1}{2}$.
Solution: By symmetry, $f_1(u) = f_0(u-1)$, thus the ML decision rule is

$$\text{Say } H_1 \text{ iff } u \geq 1$$

The false alarm probability is

$$P_{FA} = \int_1^{1+b} f_0(u)du = \frac{b}{4}$$

And by symmetry, $P_{MD} = \frac{b}{4}$.

(c) A particular transmitter sends either signal X or signal Y, with equal probability ($\pi_0 = \pi_1 = \frac{1}{2}$). The channel is noisy, therefore the receiver observes only the noisy signal Z. The receiver must determine whether H_0 or H_1 is true, with total probability of error $P_e \leq 0.1$. What is the maximum allowable value of b?
Solution: Minimum probability of error is achieved by the MAP decision rule; when $\pi_0 = \pi_1$, the MAP rule and the ML rule are the same. The total probability of error is

$$P_e = \pi_1 P_{MD} + \pi_0 P_{FA} = \frac{b}{4}$$

In order to guarantee that $P_e \leq 0.1$, therefore, we need only guarantee that $b \leq 0.4$.

4. **[Rayleigh Fading]**
A communications signal with Fourier transform $X(f)$ is transmitted through a channel with frequency response $H(f)$, generating a received signal with Fourier transform $Y(f) = H(f)X(f)$. A reverberant channel (a channel with a very large number of echoes, e.g., caused by reflection of the radio signal from nearby buildings) is well modeled by a complex Gaussian channel, i.e., $H(f) = H_R + jH_I$ where $H_R \sim \mathcal{N}(0,\sigma^2)$ and $H_I \sim \mathcal{N}(0,\sigma^2)$. Suppose for simplicity that $X(f) = 1$; then the magnitude of the received signal is

$$|Y(f)| = Y = \sqrt{H_R^2 + H_I^2}$$

(a) Find $F_Y(c)$, the CDF of the received signal.

Solution:

$$
\begin{aligned}
F_Y(c) &= \Pr\left\{H_R^2 + H_I^2 \le c^2\right\} \\
&= \int_{-c}^{c} \int_{-\sqrt{c^2-u^2}}^{\sqrt{c^2-v^2}} f_{H_R}(u) f_{H_I}(v) du dv \\
&= \int_{0}^{c} \int_{-\pi}^{\pi} \frac{1}{2\pi} e^{-\frac{r^2}{2}} r d\theta dr \\
&= \int_{0}^{c} e^{-\frac{r^2}{2}} r dr \\
&= \int_{0}^{c^2/2} e^{-w} dw \\
&= 1 - e^{-c^2/2}, \quad c \ge 0
\end{aligned}
$$

(b) Find $f_Y(u)$, the pdf of the received signal.

Solution:

$$
f_Y(u) = \begin{cases} ue^{-u^2/2} & u \ge 0 \\ 0 & \text{otherwise} \end{cases}
$$

(c) Suppose that communication fails whenever $Y \le 0.5$, i.e., whenever the received signal amplitude is less than one half of the transmitted signal amplitude. What is the probability of a communication link failure?

Solution:

$$
F_Y(0.1) = 1 - e^{-1/8} = 0.1175
$$