University of Illinois **Spring 2013**

ECE 313: Problem Set 10: Solutions
Joint Distributions, Independence

1. [**A joint pmf**]

   (a) The marginal pmfs are the column and row sums shown in the table below.

	u=0	u=1	u=2	u=3	Row sum = $p_Y(v)$
v=4	0	0.1	0.1	0.2	0.4
v=5	0.2	0	0	0	0.2
v=6	0	0.2	0.1	0.1	0.4
Column sum $p_X(u)$	0.2	0.3	0.2	0.3	

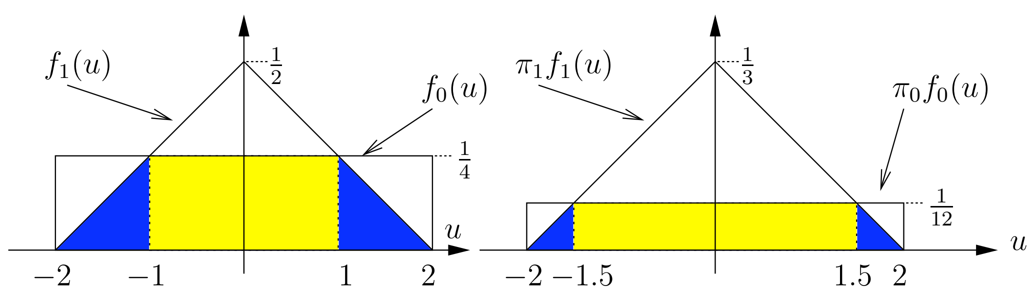
   (b) The possible values of Z are 5 through 9;
   $(p_Z(5), p_Z(6), p_Z(7), p_Z(8), p_Z(9)) = (0.3,\ 0.1,\ 0.4,\ 0.1,\ 0.1)$.

   (c) No. For example, $p_{X,Y}(0,4) = 0 \neq 0.08 = p_X(0)p_Y(4)$.

   (d) Normalizing the column for $u = 3$ yields that $p_{Y|X}(v|3)$ is equal to $\frac{2}{3}$ for $v = 4$, $\frac{1}{3}$ for $v = 6$, and zero for other values of v. Therefore, $E[Y|X = 3] = 4 \cdot \frac{2}{3} + 6 \cdot \frac{1}{3} = \frac{14}{3} = 4.666\ldots$.

2. []

   (a) The easiest way to solve this problem is to sketch the two pdfs as shown in the left-hand figure below.



   It is obvious that the maximum-likelihood decision is in favor of H_1 if $|\mathbb{X}| < 1$, and hence $x = 0$, $\eta = 1$. By inspection, we get that $P_{\text{FA}} = 2 \times \frac{1}{4} = \frac{1}{2}$ while $P_{\text{MD}} = 2 \times \left(\frac{1}{2} \times 1 \times \frac{1}{4}\right) = \frac{1}{4}$.

   The graphically-challenged can proceed as follows.
   For $-2 < u < 2$, the likelihood ratio is $\Lambda(u) = \dfrac{f_1(u)}{f_0(u)} = \dfrac{0.25(2 - |u|)}{0.25} = 2 - |u|$.
   When $\mathbb{X} = u$ is the observation, the *maximum-likelihood* decision rule decides in favor of H_1 if $\Lambda(u) > 1$. Hence $\Gamma_1 = \{u\ :\ |u| < 1\}$ and $\Gamma_0 = \{u\ :\ 1 < |u| < 2\}$, that is, the ML decision rule is that if $|\mathbb{X}| > 1$, the decision is that H_0 is the true hypothesis. Thus, we have $x = 0$, and $\eta = 1$.
   $P_{\text{FA}} = \displaystyle\int_{\Gamma_1} f_0(u)\,du = \int_{-1}^{1} \frac{1}{4} du = \frac{1}{2}.$
   $P_{\text{MD}} = \displaystyle\int_{\Gamma_0} f_1(u)\,du = 2\int_{1}^{2} \frac{1}{4}(2-u)\,du = \frac{1}{2}\left(2u - \frac{u^2}{2}\right)\Bigg|_1^2 = \frac{1}{4}.$

   (b) The probability of error of the ML decision rule is

$$P(E) = \pi_0 P_{\text{FA}} + \pi_1 P_{\text{MD}} = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{4} = \frac{1}{3}.$$

(c) Sketching $\pi_0 f_0(u)$ and $\pi_1 f_1(u)$ as in the right-hand figure above, we easily see that the MAP decision is in favor of H_1 if $|\mathbb{X}| < 1.5$, and hence $x = 0$, $\xi = 1.5$. By inspection, we get that $\pi_0 P_{\text{FA}} = 3 \times \frac{1}{12} = \frac{1}{4} = \frac{1}{3} \times \frac{3}{4}$ while $\pi_1 P_{\text{MD}} = 2 \times \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{12}\right) = \frac{1}{24} = \frac{2}{3} \times \frac{1}{16}$, that is, $P_{\text{FA}} = \frac{3}{4}$, and $P_{\text{MD}} = \frac{1}{16}$. $P(E) = \pi_0 P_{\text{FA}} + \pi_1 P_{\text{MD}} = \frac{1}{4} + \frac{1}{24} = \frac{7}{24} < \frac{1}{3}$, where $\frac{1}{3}$ is the error probability of the ML rule (with the same *a priori* probabilities) that we found in part (c).

Without using any graphical aids, we have that when $\mathbb{X} = u$ is the observation, the MAP decision rule decides in favor of H_1 if $\Lambda(u) = 2 - |u| > \pi_0/\pi_1 = 1/2$. Hence, $\Gamma_1 = \left\{u : |u| < \frac{3}{2}\right\}$ and $\Gamma_0 = \left\{u : \frac{3}{2} < |u| < 2\right\}$ for the MAP decision rule. Once again, $x = 0$ while $\xi = \frac{3}{2}$. We get

$$P_{\text{FA}} = \int_{\Gamma_1} f_0(u)\,du = \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{4} du = \frac{3}{4}.$$

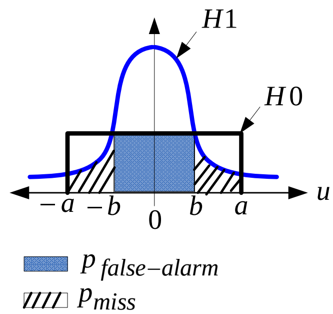
$$P_{\text{MD}} = \int_{\Gamma_0} f_1(u)\,du = 2\int_{\frac{3}{2}}^{2} \frac{1}{4}(2-u)\,du = \frac{1}{2}\left(2u - \frac{u^2}{2}\right)\Bigg|_{\frac{3}{2}}^{2} = \frac{1}{16}.$$

Hence, $P(E) = \pi_0 \cdot P_{\text{FA}} + \pi_1 \cdot P_{\text{MD}} = \dfrac{1}{3} \times \dfrac{3}{4} + \dfrac{2}{3} \times \dfrac{1}{16} = \dfrac{1}{4} + \dfrac{1}{24} = \dfrac{7}{24} < \dfrac{1}{3}.$

3. []

(a) From the figure, the pdf for H_1 is smaller than the pdf for H_0 precisely when $b < |u| < a$. Thus, the ML rule is given by:

$$\widehat{H} = \begin{cases} H_0 & b < |X| < a \\ H_1 & \text{otherwise} \end{cases}$$



(b)

(c) These probabilities are calculated as follows:

$$\begin{aligned} p_{false\ alarm} &= \frac{2b}{2a} = \frac{b}{a} \\ p_{miss} &= 2(\Phi(a) - \Phi(b)) = 2(Q(b) - Q(a)) \end{aligned}$$

(d) Given π_1, one obtains

$$\begin{aligned} \pi_0 &= 1 - \pi_1 = \frac{\sqrt{3}}{3 + \sqrt{2\pi}} \\ \frac{\pi_0}{\pi_1} &= \frac{3}{\sqrt{2\pi}} \end{aligned}$$

The LRT gives us

$$\begin{aligned} \frac{3}{\sqrt{2\pi}} e^{\frac{-u^2}{2}} &= \frac{3}{\sqrt{2\pi}} \\ e^{\frac{-u^2}{2}} = 1 \end{aligned}$$

Thus, the MAP rule is given by

$$\hat{H} = \begin{cases} H_1 & u > 0 \\ H_0 & otherwise \end{cases}$$

4. []

(a) Since $\int_{-1}^{1} f_1(u)\, du = 1$, we see that $C = 1$.

(b) The ML decision region Γ_0 is the set of all u such that $f_0(u) > f_1(u)$. This happens whenever $|u| < 0.5$.